On Some Problems for Deformations and Oscillations of Non-homogeneous Piezoelectric Elastic Beam

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By
Archili Sakevarashvili

Supervisor: Prof. George Jaiani

Ivane Javakhishvili Tbilisi State University

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Annotation

In the present work several problems on deformation and oscillation of non-homogeneous piezo-electric elastic beam is studied. The deformation is limited to be only along x_3 axis and it is assumed that all functions depend on time t and/or spatial variable x_3. Deformations and oscillations of such material was studied in case when the constitutive coefficients were constants and when they were power functions of spatial variable x_3, i.e. equal to $const. \times x_3^{\kappa}$ for some $\kappa = const. \in (0, 1)$. With given charge density (f_e) and volume force along x_3 axis (Φ_3), initial-boundary value problems were studied. Solutions for displacement vector (u_3) as well as electric (χ) and magnetic (η) potentials that arise during the deformation were written in analytic form (constant-coefficient case) or as an infinite series (coefficients given power functions of variable x_3) and absolute uniform convergence of these series were studied. Furthermore, volume force components along x_1 and x_2 axis were found for which the deformation will be limited along x_3 axis. The solutions for the case when constitutive coefficients were constants were plotted for specific material (mixture of $CoFe_2O_4$ and $BaTiO_3$) using MATLAB.

Introduction

The development of science, industry and technologies on the one hand made the possibility of constructing such new complex composed materials with different physical properties (piezoelectric, piezomagnetic, multi-component mixtures, bio-materials, meta-materials etc.) that are not found naturally on Earth. On the other hand these new materials can be used for future development of the same fields. Several examples include piezoelectric sensors for vibration control ([24]), high precision actuators ([10]), materials with higher strength and stiffness ([29]) or ones that lower energy consumption ([23][30]), production cost and size of sensors or actuators ([10][24]).

Piezoelectric materials did not come into widespread use until the World War I, when quartz was used as resonators for ultrasound sources in SONAR to detect submarines through echolocation ([31]). Although nowadays such materials can be seen in daily life even in devices such as speakers, headphones or microphones.

The increasing demand on developing new types of materials makes it necessary to describe mathematically how do they behave under the influence of some physical fields.

Direct piezoelectric effect was discovered by the brothers Jacques Curie (1856-1941) and Pierre Curie (1859-1906) ([5][6]). They discovered, that for certain materials (such as quartz, Rochelle salt, topaz and etc.) tension and compression generated voltages of opposite polarity and proportional to the applied load. The word hydroelectricity comes from Greek language and means electricity resulting from pressure. Later, Lippman ([21]) predicted existence of the converse effect based on thermodynamic principles, that was experimentally proved by the brothers Curies in 1881.

The Magnetoelectric effect was first predicted by Landay and Lifshitz in 1957 ([20]) and was later confirmed in an antiferromagnetic single crystal Cr_2O_3 ([2][9]).

The electromagnetic effects in solid bodies was studied by V. Nowacki ([37]), P. Denieva ([31]). Other examples of studies can be found in [1][8][33][34].

The governing equations for thermo-piezo-electro-magneto-elastic material with voids are given e.g. by G. Jaiani in [13]. They consist of: 1. motion equations; 2. kinematic relations; 3. constitutive equations. Constitutive equations and constants (e.g. piezoelectric and piezomagnetic coefficients, dielectric and magnetic permittivity constant, etc.) are determined by experimental studies.

Solving boundary and/or initial value problems for differential equation systems related to body deformations can be challenging for example when cusped plates are considered, i.e. such ones whose thickness on the part of the plate boundary or on the whole one vanishes. In 1955 I. Vekua published his models of elastic prismatic shells ([16]). In 1965 he offered analogous models for standard shells ([17]). Later the problem was furthermore investigated for different types of materials by G. Jaiani ([11][12][13][14][15]), N. Chinchaladze ([4][25][26][27][28]), T. Buchukuri, T. Chkadua and D. Natroshvili ([3][7][35][36]).

In the present work several problems were studied for deformations and oscillations of non-homogeneous piezoelectric elastic beam. The beam can be thought as a rectangular cuboid with constant height, length and width (generally, width and height of a beam can be variable, but in the present work they are considered constants). The problem was studied when the body deformation was limited to be only along x_3 axis and all functions where depended only on x_3 spatial variable (and on time t in case of dynamics). The main problem was to find the displacement (u_3), electric potential (χ) and magnetic potential (η) when charge density (f_e) and volume force along x_3 axis (Φ_3) were given. Finally volume forces along x_1 and x_2 axis were determined for which the deformation would be limited only along x_3 axis.

The work is organized as follows: in Section 1 the system of differential equations is derived for deformation of piezoelectric elastic material when the body deformation is limited to be only along x_3 axis; in Section 2 deformation of piezoelectric elastic material with constant constitutive coefficients is studied with static problem discussed in 2.1 and dynamic problem in 2.2; in Section 3 the problem is discussed when the constitutive coefficients are considered to be power functions of spatial variable x_3, i.e. constants equal to $const. \times x_3^\kappa$ for some $\kappa = cosnt. \in (0,1)$. The equations were reduced to linear integral equation of second kind. Using Hilbert-Schmidt theorems ([18][22][19]) the solution was written as an infinite series and absolute uniform convergence of the series was proved; in Section 4 plots of the solutions are given for the problems discussed in Section 2. Supplementary theorems that are used throughout the work are given in Appendix A; in Appendix B MATLAB codes for the plots shown in Section 4 are given; in Appendix C additional remarks are provided.

1 Preliminary Materials

1.1 System of Differential Equations

In the present work a piezoelectric elastic beam is considered ([7][13]):

$$\bar{V} := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_3 \leq L, 0 \leq x_1 \leq d, 0 \leq x_2 \leq h\} \tag{1}$$

where $L, h = const$.

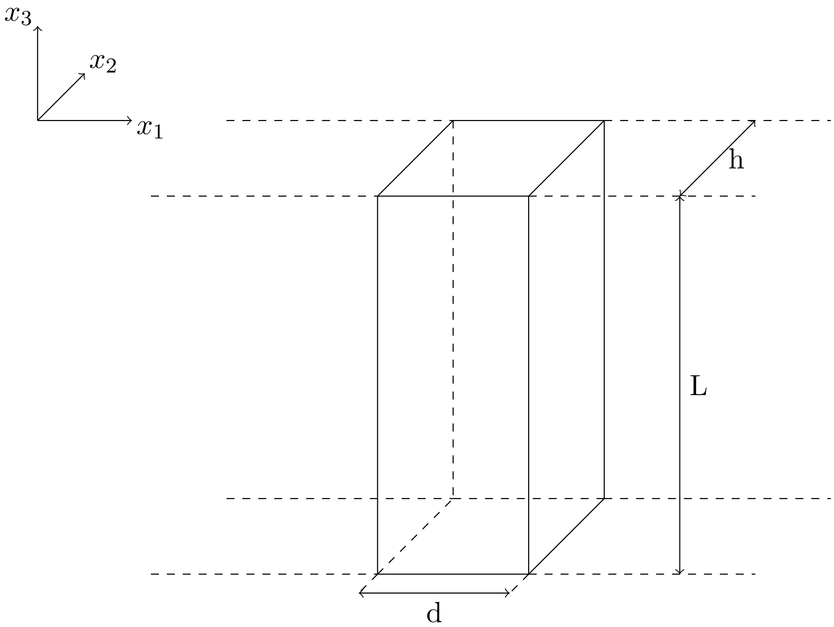

Figure 1.1: Beam given by region $\bar{V}$

Fig. 1.1 shows the region occupied be the beam given by (1). From the perspective of physics, the beam can be represented as infinite number of rectangular cuboids with height h, length L and d width, stacked along x_1 axis. In the present work the deformation is limited to be only along x_3 axis and all functions depend only on spatial variable x_3 and time t.

The governing equations for piezoelectric Kelvin-Voigt materials with voids has the following form (see e.g. [13]):

Motion Equations

$$\begin{aligned} X_{ji,j} + \Phi_i = \rho \ddot{u}_i(x_1, x_2, x_3, t), (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3, \\ t > t_0; i, j = \overline{1,3} \end{aligned} \tag{2}$$

$$D_{j,j} = f_e, \qquad B_{j,j} = 0, \qquad \Omega \times]0, T[, \qquad j = \overline{1,3} \tag{3}$$

where $X_{ij} \in C^1(\Omega)$ is the stress tensor; Φ_i are the volume force components; ρ is the mass density; $u_i \in C^2(\Omega)$ are the displacements; $f_e : \Omega \times]0, T[\to \mathbb{R}^1$ is the electric charge density; $\mathbf{D} := (D_1, D_2, D_3) : \Omega \times]0, T[\to \mathbb{R}^3$ is the electrical displacement vector; $\mathbf{B} := (B_1, B_2, B_3) : \Omega \times]0, T[\to \mathbb{R}^3$ is the magnetic induction vector. Here and in the future Einstein summation convention is used.

Kinematic Relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i,j = \overline{1,3} \tag{4}$$

where $e_{ij} \in C^1(\Omega)$ is the strain tensor.

Constitutive Equations

$$X_{ji} = X_{ij} = E_{ijkl}e_{kl} + p_{kij}\chi_{,k} + q_{kij}\eta_{,k}, \qquad i,j,k,l = \overline{1,3} \tag{5}$$

$$D_j = p_{jkl}e_{kl} - \varsigma_{jl}\chi_{,l} - \tilde{a}_{jl}\eta_{,l}, \qquad i,j,k,l = \overline{1,3} \tag{6}$$

$$B_j = q_{jkl}e_{kl} - \tilde{a}_{jl}\chi_{,l} - \xi_{jl}\eta_{,l}, \qquad i,j,k,l = \overline{1,3} \tag{7}$$

where E_{ijkl} are elastic constants (measured at constant electric and magnetic fields), $\chi : \Omega\times]0,T[\to \mathbb{R}^1$ and $\eta : \Omega\times]0,T[\to \mathbb{R}^1$ are electric and magnetic potentials, respectively; p_{kij} are piezoelectric coefficients (measured at constant magnetic field), and q_{kij} are piezomagnetic coefficients (measured at constant electric field); ς_{jl} and ξ_{jl} are dielectric permittivity coefficients (measured at constant strain and magnetic filed) and magnetic permeability coefficients (measured at constant strain and electric field), respectively; $\tilde{a}_{jl}$ are the coupling coefficients (so called magnetoelectric coefficients) connecting electric and magnetic fields (measured at constant strain) ([13][38]). It can be shown, that the constitutive coefficients $E_{ijkl}, p_{kij}, q_{kij}, \varsigma_{jl}, \tilde{a}_{jl}, \xi_{jl}$ satisfy the following symmetry relations ([37]):

$$\begin{gathered} E_{ijkl} = E_{jikl} = E_{jilk} = E_{klij}, \xi_{jl} = \xi_{lj}, \tilde{a}_{jl} = \tilde{a}_{lj} \\ p_{kij} = p_{kji}, q_{kij} = q_{kji}, \varsigma_{jl} = \varsigma_{lj} \\ i,j,k,l = \overline{1,3} \end{gathered} \tag{8}$$

In the present work the case when the body deformation is limited along x_3 axis is studied, i.e. $u_1 = u_2 = 0$. Under this consideration, if we insert (4) into (5)-(7) and the result into equations (2) and (3), and use (8) relations it will lead us to the following system of equations:

$$(E_{\alpha333}u_{3,3} + p_{3\alpha3}\chi_{,3} + q_{3\alpha3}\eta_{,3})_{,3} + \Phi_\alpha = 0, \alpha = 1,2 \tag{9}$$

$$(E_{3333}u_{3,3} + p_{333}\chi_{,3} + q_{333}\eta_{,3})_{,3} + \Phi_3 = \rho\ddot{u}_3, \tag{10}$$

$$(p_{333}u_{3,3} - \varsigma_{33}\chi_{,3} - \tilde{a}_{33}\eta_{,3})_{,3} = f_e \tag{11}$$

$$(q_{333}u_{3,3} - \tilde{a}_{33}\chi_{,3} - \xi_{33}\eta_{,3})_{,3} = 0 \tag{12}$$

Here, Φ_3, ρ and f_e are known functions and we need to solve the system of equations for u_3, χ and η. The general idea of solving this system of equations is to find functions (x_3, χ, η) from the system (10)-(12). Then (9) equations will become conditions for Φ_1 and Φ_2 components of volume force. From the perspective of physics these conditions can be interpreted as follows: the condition which Φ_1 and Φ_2 must satisfy in order to have deformation only along x_3 axis.

Note that equations (9) and (10) are obtained from motion equation (2), whereas equations (11) and (12) from ones given by (3).

2 Constitutive Coefficients are Constants

In this section the case when: $E_{i333}, p_{333}, q_{333}, \varsigma_{33}, \tilde{a}_{33}, \xi_{33} = cosnt., i = \overline{1,3}$ is studied. In this case, equations (9)-(12) becomes:

$$E_{\alpha 333} u_{3,33} + p_{3\alpha 3} \chi_{,33} + q_{3\alpha 3} \eta_{,33} + \Phi_\alpha = 0, \alpha = 1,2; k = \overline{1,3} \tag{13}$$

$$E_{3333} u_{3,33} + p_{333} \chi_{,33} + q_{333} \eta_{,33} + \Phi_3 = \rho \ddot{u}_3, \tag{14}$$

$$p_{333} u_{3,33} - \varsigma_{33} \chi_{,33} - \tilde{a}_{33} \eta_{,33} = f_e \tag{15}$$

$$q_{333} u_{3,33} - \tilde{a}_{33} \chi_{,33} - \xi_{33} \eta_{,33} = 0 \tag{16}$$

If we consider the system of equations (14)-(16) as an algebraic system for ($u_{3,33}$, $\chi_{,33}$,$\eta_{,33}$), to solve the system we have to restrict the following determinant to be nonzero:

$$D \equiv \begin{vmatrix} E_{3333} & p_{333} & q_{333} \\ p_{333} & -\varsigma_{33} & -\tilde{a}_{33} \\ q_{333} & -\tilde{a}_{33} & -\xi_{33} \end{vmatrix} \neq 0 \tag{17}$$

2.1 Static Problem

If we consider the case, when all functions depend only on spatial variable x_3 and are not functions of time, the right hand side of (14) will become zero and we get:

$$E_{3333} u_{3,33} + p_{333} \chi_{,33} + q_{333} \eta_{,33} + \Phi_3 = 0 \tag{18}$$

The rest of equations remain the same as given by (13), (15) and (16).

Let $u_3(x_3) \in C^2([0, L])$ and consider the following boundary conditions:

$$\begin{gathered} u_3(0) = a_0, \chi(0) = b_0, \eta(0) = d_0, \\ u_3(L) = a_1, \chi(L) = b_1, \eta(L) = d_1 \end{gathered} \tag{19}$$

If we multiply (15) by ξ_{33}, (16) by $\tilde{a}_{33}$ and subtract one from another, we get $\chi_{,33}$ as a function of $u_{3,33}$:

$$\chi_{,33} = \frac{\xi_{33} p_{333} - \tilde{a}_{33} q_{333}}{\xi_{33} \varsigma_{33} - \tilde{a}_{33}^2} u_{3,33} - \frac{\xi_{33}}{\xi_{33} \varsigma_{33} - \tilde{a}_{33}^2} f_e \tag{20}$$

Similarly, from (15) and (16) we get:

$$\eta_{,33} = \frac{\tilde{a}_{33} p_{333} - \varsigma_{33} q_{333}}{\tilde{a}_{33}^2 - \xi_{33} \varsigma_{33}} u_{3,33} - \frac{\tilde{a}_{33}}{\tilde{a}_{33}^2 - \xi_{33} \varsigma_{33}} f_e \tag{21}$$

Together with (18), (20)-(21) gives:

$$L_1 u_{3,33} - L_2 f_e + \Phi_3 = 0 \tag{22}$$

where

$$L_1 = E_{3333} + \frac{\xi_{33} p_{333}^2 - 2\tilde{a}_{33} p_{333} q_{333} + \varsigma_{33} q_{333}^2}{\xi_{33}\varsigma_{33} - \tilde{a}_{33}^2}$$

$$L_2 = \frac{\xi_{33} p_{333} - \tilde{a}_{33} q_{333}}{\xi_{33}\varsigma_{33} - \tilde{a}_{33}^2}$$

After integration of (22) twice from 0 to x_3 we get general solution for u_3:

$$u_3 = -\frac{1}{L_1}\int_0^{x_3} (x_3 - y)\left(\Phi_3(y) - L_2 f_e(y)\right) dy + c_1 x_3 + c_2 \tag{23}$$

Using the boundary conditions (19) we get:

$$\begin{gathered} c_1 = \frac{1}{L}\left[L_1(a_1 - a_0) - \int_0^L (L - y)\left(L_2 f_e(y) - \Phi_3(y)\right)\right], \\ c_2 = L_1 a_0 \end{gathered} \tag{24}$$

Then from (23) and (24):

$$u_3 = L_1 a_0 + \frac{x_3}{L}(a_1 - a_0) + \frac{1}{L_1}\int_0^L K(x_3, y)\left(\Phi_3(y) - L_2 f_e(y)\right) dy \tag{25}$$

where

$$K(x_3, y) = K(y, x_3) = \begin{cases} y\left(1 - \frac{x_3}{L}\right) & 0 \le y \le x_3 \\ x_3\left(1 - \frac{y}{L}\right) & x_3 \le y \le L \end{cases} \tag{26}$$

Similarly, after integration of (20) and (21) twice from 0 to x_3 we get:

$$\chi(x_3) = L_{23} u_3 - L_3 \int_0^{x_3} (x_3 - y) f_e(y) dy + c_3 x_3 + c_4 \tag{27}$$

$$\eta(x_3) = L_4 u_3 - L_5 \int_0^{x_3} (x_3 - y) f_e(y) dy + c_5 x_3 + c_6 \tag{28}$$

where

$$L_{2i} = \frac{p_{3i3}\xi_{33} - q_{3i3}\tilde{a}_{33}}{\xi_{33}\varsigma_{33} - (\tilde{a}_{33})^2}, \qquad i = \overline{1,3} \tag{29}$$

$$L_3 = \frac{\xi_{33}}{\xi_{33}\varsigma_{33} - (\tilde{a}_{33})^2} \tag{30}$$

$$L_4 = \frac{p_{333}\tilde{a}_{33} - \varsigma_{33} q_{333}}{(\tilde{a}_{33})^2 - \varsigma_{33}\xi_{33}} \tag{31}$$

$$L_5 = \frac{\tilde{a}_{33}}{(\tilde{a}_{33})^2 - \varsigma_{33}\xi_{33}} \tag{32}$$

Considering the boundary conditions (19):

$$
\begin{gathered}
c_3 = \frac{1}{L}\left(b_1 - b_0 + L_{23}(a_0 - a_1)\int_0^L (L-y) f_e(y)dy\right) \\
c_4 = b_0 - L_{23}a_0 \\
c_5 = \frac{1}{L}\left(d_1 - d_0 + L_4(a_0 - a_1)\int_0^L (L-y) f_e(y)dy\right) \\
c_6 = d_0 - L_4 a_0
\end{gathered} \tag{33}
$$

Then from (27), (28) and (33) we get final expressions for χ and η:

$$
\chi(x_3) = L_{23}u_3(x_3) + \frac{x_3}{L}(b_1 - b_0 + L_{23}(a_0 - a_1)) + L_3\int_0^L K(x_3, y) f_e(y)dy + b_0 - L_{23}a_0
$$

$$
\eta(x_3) = L_4 u_3(x_3) + \frac{x_3}{L}(d_1 - d_0 + L_4(a_0 - a_1)) + L_5\int_0^L K(x_3, y) f_e(y)dy + d_0 - L_4 a_0
$$

Finally, from (13) and (22) we get the following expressions for Φ_1 and Φ_2:

$$
\Phi_\alpha(x_3) = \left(L_{2\alpha} - \frac{L_{1\alpha}}{L_{13}}L_{23}\right) f_e(x_3) + \frac{L_{1\alpha}}{L_{13}}\Phi_3(x_3), \qquad \alpha = 1, 2
$$

where

$$
L_{1i} = E_{i333} + \frac{p_{3i3}(\xi_{33}p_{333} - \tilde{a}_{33}q_{333})}{\xi_{33}\varsigma_{33} - (\tilde{a}_{33})^2} + \frac{q_{3i3}(\tilde{a}_{33}p_{333} - \varsigma_{33}q_{333})}{(\tilde{a}_{33})^2 - \varsigma_{33}\xi_{33}}, \quad i = \overline{1,3} \tag{34}
$$

Note, that $L_{13} = L_1$ and $L_{23} = L_2$.

2.2 Dynamic Problem

In this section the case is studied, when constitutive coefficients are constants and all functions depend on both - the spatial variable x_3 and time t. If we start to solve the (14)-(16) system in the similar way as we did in Section 2.1, we sill get the equation similar to (22), but with non-zero right-hand side:

$$
L_1 u_{3,33}(x_3, t) - L_2 f_e(x_3, t) + \Phi_3(x_3, t) = \rho \ddot{u}_3(x_3, t) \tag{35}
$$

Let the functions have the following form:

$$
\begin{gathered}
u_3(x_3, t) = u_3^0(x_3)e^{-i\omega t}, f_e(x_3, t) = f_e^0(x_3)e^{-i\omega t}, \Phi_3(x_3, t) = \Phi_3^0(x_3)e^{-i\omega t} \\
\eta = \eta^0(x_3)e^{-i\omega t}, \quad \chi = \chi^0(x_3)e^{-i\omega t}
\end{gathered} \tag{36}
$$

where $u_3^0(x_3) \in C^2([0, L])$ and consider the following boundary conditions:

$$
\begin{gathered}
u_3(0, t) = a^* e^{-i\omega t}, \chi(0, t) = a_0 e^{-i\omega t}, \eta(0, t) = b_0 e^{-i\omega t} \\
u_3(L, t) = b^* e^{-i\omega t}, \chi(L, t) = a_L e^{-i\omega t}, \eta(L, t) = b_L e^{-i\omega t}
\end{gathered} \tag{37}
$$

From (35) and (36) we get:

$$L_1 u^0_{3,33}(x_3) + \rho\omega^2 u^0_3(x_3) = L_2 f^0_e(x_3) - \Phi^0_3(x_3) \tag{38}$$

Furthermore, let us denote:

$$g(x_3) \equiv \frac{L_2 f^0_e(x_3)}{L_1} - \frac{\Phi^0_3(x_3)}{L_1}$$

and

$$\omega_0^2 \equiv \frac{\rho\omega^2}{|L_1|}$$

Then equation (38) becomes:

$$u^0_{3,33}(x_3) + sign(L_1)\frac{\rho\omega^2}{|L_1|} u^0_3(x_3) = g(x_3) \tag{39}$$

where

$$sign(L_1) = \begin{cases} 1 & L_1 > 0 \\ -1 & L_1 < 0 \end{cases}$$

We solve (39) using the method of variation of parameter ([39]).

Case 1: $L_1 > 0$

In the case when $L_1 > 0$ it can be shown, that the solution of (39) has the following form:

$$\begin{aligned} u^0_3(x_3) = c_1 e^{-i\omega_0 x_3} + c_2 e^{i\omega_0 x_3} - e^{-i\omega_0 x_3} \int_0^{x_3} \frac{e^{i\omega_0 \xi} g(\xi)}{W(e^{-i\omega_0 \xi}, e^{i\omega_0 \xi})(\xi)} d\xi \\ + e^{i\omega_0 x_3} \int_0^{x_3} \frac{e^{-i\omega_0 \xi} g(\xi)}{W(e^{-i\omega_0 \xi}, e^{i\omega_0 \xi})(\xi)} d\xi \end{aligned} \tag{40}$$

where $c_1 e^{-i\omega_0 x_3} + c_2 e^{i\omega_0 x_3}$ is general solution of the corresponding homogeneous equation of (39) and $W(e^{-i\omega_0 x_3}, e^{i\omega_0 x_3})$ is the Wronskian of $e^{-i\omega_0 x_3}$ and $e^{i\omega_0 x_3}$:

$$W(e^{-i\omega_0 x_3}, e^{i\omega_0 x_3}) = 2i\omega_0 e^{i\omega_0 x_3} e^{-i\omega_0 x_3}$$

From (40) using the boundary conditions (37) we obtain the following expressions for c_1 and c_2:

$$\begin{aligned} c_1 &= \frac{1}{2isin(\omega_0 L)} \left(a^* e^{i\omega_0 L} - b^* + \frac{1}{\omega_0} \int_0^L g(y) sin(\omega_0 (L-y)) dy \right) \\ c_2 &= \frac{-1}{2isin(\omega_0 L)} \left(a^* e^{-i\omega_0 L} - b^* + \frac{1}{\omega_0} \int_0^L g(y) sin(\omega_0 (L-y)) dy \right) \end{aligned}$$

Substitution of these coefficients into (40) and using the relation $e^{i\theta} = cos(\theta) + isin(\theta)$ we get:

$$u^0_3(x_3) = Acos(\omega_0 x_3) + Bsin(\omega_0 x_3) + \frac{1}{\omega_0} \int_0^{x_3} g(\xi) sin\left(\omega_0 (x_3 - \xi)\right) d\xi \tag{41}$$

where

$$A = a^*$$

and

$$B = -a^* ctg(\omega_0 L) + \frac{b^*}{sin(\omega_0 L)} - \frac{1}{\omega_0 sin(\omega_0 L)} \int_0^L g(\xi) sin\left(\omega_0 (L-\xi)\right) d\xi$$

To simplify (41) let us denote:

$$R = \sqrt{A^2 + B^2}$$

then equation (41) can be written as:

$$\begin{aligned} u_3^0(x_3) = R\left(\frac{A}{\sqrt{A^2+B^2}} cos(\omega_0 x_3) + \frac{B}{\sqrt{A^2+B^2}} sin(\omega_0 x_3)\right) \\ + \frac{1}{\omega_0} \int_0^{x_3} g(\xi) sin\left(\omega_0 (x_3 - \xi)\right) d\xi \end{aligned} \tag{42}$$

Further, if we denote:

$$\frac{A}{\sqrt{A^2+B^2}} = cos(\varphi)$$

we get our final expression for $u_3^0(x_3)$:

$$u_3^0(x_3) = Rcos(\omega_0 x_3 - \varphi) + \frac{1}{\omega_0} \int_0^{x_3} g(\xi) sin\left(\omega_0 (x_3 - \xi)\right) d\xi \tag{43}$$

Case 2: $L_1 < 0$

Similarly, if $L_1 < 0$, general expression for $u_3^0(x_3)$ can be written as follows:

$$\begin{aligned} u_3^0(x_3) = c_1 e^{-\omega_0 x_3} + c_2 e^{\omega_0 x_3} - e^{-\omega_0 x_3} \int_0^{x_3} \frac{e^{\omega_0 \xi} g(\xi)}{W(e^{-\omega_0 \xi}, e^{\omega_0 \xi})(\xi)} d\xi \\ + e^{\omega_0 x_3} \int_0^{x_3} \frac{e^{-\omega_0 \xi} g(\xi)}{W(e^{-\omega_0 \xi}, e^{\omega_0 \xi})(\xi)} d\xi \end{aligned} \tag{44}$$

where $c_1 e^{-\omega_0 x_3} + c_2 e^{\omega_0 x_3}$ are general solutions of the corresponding homogeneous equation of (39) and $W(e^{-\omega_0 x_3}, e^{\omega_0 x_3})$ is the Wronskian of $e^{-\omega_0 x_3}$ and $e^{\omega_0 x_3}$:

$$W(e^{-\omega_0 x_3}, e^{\omega_0 x_3}) = 2\omega_0 e^{-\omega_0 x_3} e^{\omega_0 x_3}$$

Using boundary conditions (37) we get:

$$c_1 = \frac{1}{e^{\omega_0 L} - e^{-\omega_0 L}} \left(a^* e^{\omega_0 L} - b^* + \frac{1}{2\omega_0} \int_0^L g(\xi)(e^{\omega_0 (L-\xi)} - e^{-\omega_0 (L-\xi)}) d\xi\right)$$

$$c_2 = \frac{-1}{e^{\omega_0 L} - e^{-\omega_0 L}} \left(a^* e^{-\omega_0 L} - b^* + \frac{1}{2\omega_0} \int_0^L g(\xi)(e^{\omega_0 (L-\xi)} - e^{-\omega_0 (L-\xi)}) d\xi\right)$$

Considering the last expressions we get the following expression for $u_3^0(x_3)$:

$$u_3^0(x_3) = Ae^{-\omega_0 x_3} + Be^{\omega_0 x_3} + \frac{1}{2\omega_0} \int_0^{x_3} g(\xi)(e^{\omega_0 (x_3-\xi)} - e^{-\omega_0 (x_3-\xi)}) d\xi \tag{45}$$

where

$$A = \frac{1}{e^{\omega_0 L} - e^{-\omega_0 L}} \left(a^* e^{\omega_0 L} - b^* + \frac{1}{2\omega_0} \int_0^L g(\xi)(e^{\omega_0(L-\xi)} - e^{-\omega_0(L-\xi)}) d\xi \right)$$

and

$$B = \frac{1}{e^{-\omega_0 L} - e^{\omega_0 L}} \left(a^* e^{-\omega_0 L} - b^* + \frac{1}{2\omega_0} \int_0^L g(\xi)(e^{\omega_0(L-\xi)} - e^{-\omega_0(L-\xi)}) d\xi \right)$$

The following equation summarizes the results obtained for $u_3^0(x_3)$ in the two cases above:

$$u_3^0(x_3) = \begin{cases} Rcos(\omega_0 x_3 - \phi) + \frac{1}{\omega_0} \int_0^{x_3} g(\xi) sin\left(\omega_0(x_3 - \xi)\right) d\xi & L_1 > 0 \\ Ae^{-\omega_0 x_3} + Be^{\omega_0 x_3} + \frac{1}{2\omega_0} \int_0^{x_3} g(\xi)(e^{\omega_0(x_3-\xi)} - e^{-\omega_0(x_3-\xi)}) d\xi & L_1 < 0 \end{cases} \tag{46}$$

It can be easily shown, that (46) satisfies equation (39) and boundary conditions (37).

We can now easily get expression for $u_3(x_3, t)$ from (46) and (36).

Next, from (15) and (16) we will obtain the same expressions for $\eta^0_{,33}$ and $\chi^0_{,33}$ as we obtained in (20) and (21) for $\eta_{,33}$ and $\chi_{,33}$. Thus, if we solve these differential equations step-by-step in the similar way, using boundary the conditions (37) we obtain the following expressions for $\eta^0(x_3)$ and $\chi^0(x_3)$:

$$\eta^0(x_3) = L_4 u_3 + k_1 x_3 + k_2 + L_5 \int_0^L K(x_3, y) f_e(y) dy$$
$$\chi^0(x_3) = L_{23} u_3 + k_3 x_3 + k_4 + L_3 \int_0^L K(x_3, y) f_e(y) dy$$

where $K(x_3, y)$ is given by (26), the constants L_{23}, L_3, L_4, L_5 by (29)-(32) and

$$k_1 = \frac{1}{L}[L_4(a^* - b^*) + b_L - b_0]$$
$$k_2 = b_0 - L_4 a^*$$
$$k_3 = \frac{1}{L}[L_{23}(a^* - b^*) + a_L - a_0]$$
$$k_4 = a_0 - L_3 a^*$$

Finally, from (13) we get the conditions for Φ_1 and Φ_2 that, as it was stated earlier, restricts the deformation to be only along x_3 axis:

$$\Phi_\alpha(x_3) = -L_{1\alpha} u_{3,33}(x_3) + L_{2\alpha} f_e(x_3), \quad \alpha = 1, 2$$

where $L_{1\alpha}$ is defined by (34).

3 Constitutive Coefficients are Power Functions of Spatial Variable x_3

In the following section we consider the case when constitutive coefficients are power functions of spatial variable x_3:

$$\begin{aligned} E_{i333} = E^0_{i333}x_3^\kappa, p_{3i3} = p^0_{3i3}x_3^\kappa, q_{3i3} = q^0_{3i3}x_3^\kappa, \varsigma_{33} = \varsigma^0_{33}x_3^\kappa, \\ \xi_{33} = \xi^0_{33}x_3^\kappa, \tilde{a}_{33} = \tilde{a}^0_{33}x_3^\kappa, \qquad i = \overline{1,3} \end{aligned} \tag{47}$$

where $E^0_{i333}, p^0_{3i3}, q^0_{3i3}, \varsigma^0_{33}, \xi^0_{33}, \tilde{a}^0_{33}, \kappa = const, i = \overline{1,3}$ and $\kappa > 0$.

From (47) the system of equations (10)-(12) becomes:

$$(E^0_{3333}x_3^\kappa u_{3,3} + p^0_{333}x_3^\kappa \chi_{,3} + q^0_{3333}x_3^\kappa \eta_{,3})_{,3} + \Phi_3 = \rho\ddot{u}_3 \tag{48}$$

$$(p^0_{333}x_3^\kappa u_{3,3} - \varsigma^0_{33}x_3^\kappa \chi_{,3} - \tilde{a}^0_{33}x_3^\kappa \eta_{,3})_{,3} = f_e \tag{49}$$

$$(q^0_{3333}x_3^\kappa u_{3,3} - \tilde{a}^0_{33}x_3^\kappa \chi_{,3} - \xi^0_{33}x_3^\kappa \eta_{,3})_{,3} = 0 \tag{50}$$

If we consider equations (48)-(50) as algebraic system for $(x_3^\kappa u_{3,3})_{,3}$, $(x_3^\kappa \chi_{,3})_{,3}$ and $(x_3^\kappa \eta_{,3})_{,3}$, then it is evident, that to solve the system we have to require the following condition:

$$D \equiv \begin{vmatrix} E^0_{3333} & p^0_{333} & q^0_{3333} \\ p^0_{333} & -\varsigma^0_{33} & -\tilde{a}^0_{33} \\ q^0_{3333} & -\tilde{a}^0_{33} & -\xi^0_{33} \end{vmatrix} \neq 0 \tag{51}$$

Later in the work (see lemma 3.4) it is shown that the (48)-(50) system has physically valid solution iff

$$D > 0 \tag{52}$$

3.1 (48)-(50) Solution of The System of Differential Equations

Let the following conditions fulfill:

$$u_3(\cdot, t) \in C^2(]0, L[) \cap C([0, L])$$

$$u_3(x_3, \cdot) \in C^2(t > 0) \cap C^1(t \geq 0), u_3(x_3, t) \in C(0 \leq x_3 \leq L, t \geq 0)$$

Furthermore let $\kappa < 1$ and consider the following homogeneous boundary conditions:

$$u_3(0, t) = u_3(L, t) = \xi(0, t) = \xi(L, t) = \eta(0, t) = \eta(L, t) = 0 \tag{53}$$

and non-homogeneous initial conditions:

$$u_3(x_3, 0) = \varphi_1(x_3) \tag{54}$$

$$\dot{u}_3(x_3, 0) = \varphi_2(x_3) \tag{55}$$

Integration of (48) from L to x_3, dividing both sides of the resulted equation by x_3^κ and integration of the result a second time from L to x_3 gives us the following general equation:

$$\begin{aligned} E^0_{3333}u_3 + p^0_{333}\chi + q^0_{333}\eta - \frac{\rho}{1-\kappa}\int_L^{x_3}(x_3^{1-\kappa} - y^{1-\kappa})\ddot{u}_3(y,t)dy \\ = -\frac{1}{1-\kappa}\int_L^{x_3}(x_3^{1-\kappa} - y^{1-\kappa})\Phi_3(y)dy + \frac{c_{11}}{1-\kappa}(x_3^{1-\kappa} - L^{1-\kappa}) + c_{12} \end{aligned} \tag{56}$$

Then using boundary conditions (53):

$$\begin{gathered} c_{11} = \frac{1}{L^{1-\kappa}}\left[\rho\int_0^L y^{1-\kappa}\ddot{u}_3(y,t)dy - \int_0^L y^{1-\kappa}\Phi_3(y)dy\right] \\ c_{12} = 0 \end{gathered} \tag{57}$$

Considering the last result we get:

$$\begin{aligned} E^0_{3333}u_3(x_3,t) + p^0_{333}\chi(x_3,t) + q^0_{333}\eta(x_3,t) - \frac{\rho}{1-\kappa}\int_L^{x_3}(x_3^{1-\kappa} - y^{1-\kappa})\ddot{u}_3(y,t)dy \\ = -\frac{1}{1-\kappa}\int_L^{x_3}(x_3^{1-\kappa} - y^{1-\kappa})\Phi_3(y,t)dy \\ + \frac{x_3^{1-\kappa} - L^{1-\kappa}}{(1-\kappa)L^{1-\kappa}}\left(\rho\int_0^L y^{1-\kappa}\ddot{u}_3(y,t)dy - \int_0^L y^{1-\kappa}\Phi_3(y,t)dy\right) \end{aligned} \tag{58}$$

Similarly, from (49) and (50) we can get general solutions for χ and η as functions of u_3:

$$\begin{aligned} \chi(x_3,t) = \frac{\xi^0_{33}p^0_{333} - \tilde{a}^0_{33}q^0_{333}}{\xi^0_{33}\varsigma^0_{33} - (\tilde{a}^0_{33})^2}u_3(x_3,t) - \frac{1}{1-\kappa}\frac{\xi^0_{33}}{\xi^0_{33}\varsigma^0_{33} - (\tilde{a}^0_{33})^2}\int_L^{x_3}(x_3^{1-\kappa} - y^{1-\kappa})f_e(y)dy \\ - \frac{1}{\xi^0_{33}\varsigma^0_{33} - (\tilde{a}^0_{33})^2}\frac{c_{21}}{1-\kappa}(x_3^{1-\kappa} - L^{1-\kappa}) - \frac{c_{22}}{\xi^0_{33}\varsigma^0_{33} - (\tilde{a}^0_{33})^2} \end{aligned} \tag{59}$$

$$\begin{aligned} \eta(x_3,t) = \frac{\tilde{a}^0_{33}p^0_{333} - \varsigma^0_{33}q^0_{333}}{(\tilde{a}^0_{33})^2 - \xi^0_{33}\varsigma^0_{33}}u_3(x_3,t) - \frac{1}{1-\kappa}\frac{\tilde{a}^0_{33}}{(\tilde{a}^0_{33})^2 - \xi^0_{33}\varsigma^0_{33}}\int_L^{x_3}(x_3^{1-\kappa} - y^{1-\kappa})f_e(y)dy \\ - \frac{1}{(\tilde{a}^0_{33})^2 - \xi^0_{33}\varsigma^0_{33}}\frac{c_{31}}{1-\kappa}(x_3^{1-\kappa} - L^{1-\kappa}) - \frac{c_{32}}{\xi^0_{33}\varsigma^0_{33} - (\tilde{a}^0_{33})^2} \end{aligned} \tag{60}$$

Using boundary conditions (53) from (59) and (60) we get:

$$\begin{gathered} c_{21} = \frac{\xi^0_{33}}{L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y)dy \\ c_{22} = 0 \\ c_{31} = \frac{\tilde{a}^0_{33}}{L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y)dy \\ c_{32} = 0 \end{gathered} \tag{61}$$

Finally, using (61) we get:

$$
\begin{aligned}
\chi(x_3,t) = \frac{1}{\xi_{33}^0 \varsigma_{33}^0 - (\tilde{a}_{33}^0)^2} \times \\
\Bigg[(\xi_{33}^0 p_{333}^0 - \tilde{a}_{33}^0 q_{3333}^0) u_3(x_3,t) - \frac{\xi_{33}^0}{1-\kappa} \int_L^{x_3} (x_3^{1-\kappa} - y^{1-\kappa}) f_e(y,t) dy \\
- \frac{\xi_{33}^0 (x_3^{1-\kappa} - L^{1-\kappa})}{(1-\kappa) L^{1-\kappa}} \int_0^L y^{1-\kappa} f_e(y,t) dy \Bigg]
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
\eta(x_3,t) = \frac{-1}{\xi_{33}^0 \varsigma_{33}^0 - (\tilde{a}_{33}^0)^2} \times \\
\Bigg[(\tilde{a}_{33}^0 p_{333}^0 - \varsigma_{33}^0 q_{3333}^0) u_3(x_3,t) - \frac{\tilde{a}_{33}^0}{1-\kappa} \int_L^{x_3} (x_3^{1-\kappa} - y^{1-\kappa}) f_e(y,t) dy \\
- \frac{\tilde{a}_{33}^0 (x_3^{1-\kappa} - L^{1-\kappa})}{(1-\kappa) L^{1-\kappa}} \int_0^L y^{1-\kappa} f_e(y,t) dy \Bigg]
\end{aligned}
\tag{63}
$$

If we substitute (62) and (63) into (58) and make some transformations we obtain:

$$
\begin{aligned}
u_3(x_3,t) + \rho \int_0^L K(x_3,y) \ddot{u}_3(y,t) dy = -L_2 \int_0^L K(x_3,y) f_e(y,t) dy \\
+ \int_0^L K(x_3,y) \Phi_3(y,t) dy
\end{aligned}
\tag{64}
$$

where

$$
K(x_3,y) = \frac{1}{(1-\kappa) L_1 L^{1-\kappa}} \times \begin{cases} y^{1-\kappa}(L^{1-\kappa} - x_3^{1-\kappa}) & 0 \le y \le x_3 \\ x_3^{1-\kappa}(L^{1-\kappa} - y^{1-\kappa}) & x_3 \le y \le L \end{cases}
\tag{65}
$$

$$
L_1 = E_{3333}^0 + \frac{(p_{333}^0)^2 \xi_{33}^0 - 2 p_{333}^0 q_{3333}^0 \tilde{a}_{33}^0 + (q_{3333}^0)^2 \varsigma_{33}^0}{\xi_{33}^0 \varsigma_{33}^0 - (\tilde{a}_{33}^0)^2}
\tag{66}
$$

$$
L_2 = \frac{p_{333}^0 \xi_{33}^0 - q_{3333}^0 \tilde{a}_{33}^0}{\xi_{33}^0 \varsigma_{33}^0 - (\tilde{a}_{33}^0)^2}
\tag{67}
$$

As the following lemma shows, it can be shown that $K(x_3, y)$ is symmetric:

Lemma 3.1. *$K(x_3, y)$ defined by (65) is symmetric, i.e.:*

$$
K(x_3, y) = K(y, x_3)
$$

Proof. Let $z_1, z_2 \in [0, L]$. Then by direct substitution we get:

$$
K(z_1, z_2) = K(z_2, z_1)
$$

□

Additionally, according by Theorem A.5 all of the eigenvalues of $K(x, t)$ are real.

Using (62) and (63) we obtain:

$$\begin{aligned}\chi_{,3}(x_3,t)x_3^{\kappa} = \frac{1}{\xi_{33}^0\varsigma_{33}^0-(\tilde{a}_{33}^0)^2}&\Big[(p_{333}^0\xi_{33}^0-q_{3333}^0\tilde{a}_{33}^0)u_{3,3}(x_3,t)x_3^{\kappa}\\ &+\xi_{33}^0\int_{x_3}^{L}f_e(y,t)dy-\frac{\xi_{33}^0}{L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y,t)dy\Big]\end{aligned} \tag{68}$$

$$\begin{aligned}\eta_{,3}(x_3,t)x_3^{\kappa} = -\frac{1}{\xi_{33}^0\varsigma_{33}^0-(\tilde{a}_{33}^0)^2}&\Big[(\tilde{a}_{33}^0p_{333}^0-\varsigma_{33}^0q_{3333}^0)u_{3,3}(x_3,t)x_3^{\kappa}\\ &+\tilde{a}_{33}^0\int_{x_3}^{L}f_e(y,t)dy-\frac{\tilde{a}_{33}^0}{L^{1-\kappa}}\int_0^L y^{1-\kappa}f_e(y,t)dy\Big]\end{aligned} \tag{69}$$

Substitution of (68)-(69) into (48) gives us the following equation:

$$[L_1u_{3,3}x_3^{\kappa}]_{,3}-\rho\ddot{u}_3 = F(x_3,t) \tag{70}$$

where

$$F(x_3,t)\equiv L_2f_e(x_3,t)-\Phi_3(x_3,t) \tag{71}$$

Let us firstly assume that $f_e(x_3,t)\equiv 0$ and $\Phi_3(x_3,t)\equiv 0$ for all $x_3\in[0,L]$ and $t>0$. Thus for $F(x_3,t)$ defined by (71):

$$F(x_3,t)\equiv 0 \tag{72}$$

If we search for $u_3(x_3,t)$ in the following form we have:

$$u_3(x_3,t)=X(x_3)T(t) \tag{73}$$

then from (70), (72) and (73) we get:

$$\frac{\ddot{T}(t)}{T(t)}=\frac{L_1(X_{,3}(x_3)x_3^{\kappa})_{,3}}{\rho X(x_3)}=-\lambda^2=const. \tag{74}$$

Using the boundary conditions (53) from (73):

$$X(0)=X(L)=0 \tag{75}$$

Next, from (64) considering the equations (73) and (74) we get:

$$\frac{\ddot{T}(t)}{T(t)}=-\lambda^2=-\frac{X(x_3)}{\rho\int_0^L K(x_3,y)X(y)dy} \tag{76}$$

and

$$X(x_3)=\lambda^2\rho\int_0^L K(x_3,y)X(y)dy \tag{77}$$

Let us prove the following two lemmas:

Lemma 3.2. *Number of λ_n^2 eigenvalues of the equation (77) is not finite.*

Proof. Assume, for the sake of contradiction, that the number of λ_n^2 is finite, and $n = \overline{1,m}$. Then $K(x_3, y)$ can be written as (see Theorem A.2):

$$K(x_3, y) = \sum_{n=1}^{m} \frac{X_n(x_3)X_n(y)}{\lambda_n^2}$$

where $X_n(x_3) \in C^2(]0, L[)$. Thus

$$K(x_3, y) \in C^2(]0, L[) \tag{78}$$

Then

$$K'(x_3, y)\big|_{y \to x-} - K'(x_3, y)\big|_{y \to x_3+} = -\frac{x_3^{-\kappa}}{L_1}$$

i.e. $K(x_3, y) \notin C^2(]0, L[)$ that contradicts with (78). ☐

Lemma 3.3. *If L_1 given by (66) is positive than the solution of the problem is oscillatory.*

Proof. From (74) we have:

$$X(x_3) = -\frac{L_1}{\lambda^2 \rho}(X_{,3}(x_3)x_3^{\kappa})_{,3} \tag{79}$$

Let (without loss of generality, see Theorem A.4) $X_n(x_3)$ be orthonormalized eigenfunctions of (79), then

$$\lambda_n^2 X_n(x_3) = -\frac{L_1}{\rho}(X_{n,3}(x_3)x_3^{\kappa})_{,3}$$

If we multiply both sides of the last expression by $X_n(x_3)$ and integrate from 0 to L, we get:

$$\lambda_n^2 = -\frac{L_1}{\rho}\int_0^L X_n(x_3)(X_{n,3}(x_3)x_3^{\kappa})_{,3}dx_3 = \frac{L_1}{\rho}\int_0^L (X_{n,3}x_3^{\kappa/2})^2 dx_3$$

☐

Now we can prove the following lemma:

Lemma 3.4. *The solution of the system of equations (48)-(50) is oscillatory iff condition (52) is fulfilled. Condition (52) is equivalent to the following conditions: (a) $L_1 > 0$, (b) $\lambda_n^2 > 0$ for $n \in N$.*

Proof. As we proved in Lemma 3.3, the solution is oscillatory iff $L_1 > 0$.

On the other hand, from (51) and (66) we have:

$$D = \frac{L_1}{\xi_{33}^0 \varsigma_{33}^0 - (\tilde{a}_{33}^0)^2}$$

It can be shown ([7][37]) that $\xi^0_{33}\varsigma^0_{33} - (\tilde{a}^0_{33})^2 > 0$. So we conclude that D has the same sign as L_1.

This conclues our proof. $\square$

Using the result of Lemma 3.4 the solutions of (76) for eigenfunctions $T_n(t)$ with corresponding eigenvalues λ_n^2 are:

$$T_n(t) = b_1^n sin(\lambda_n t) + b_2^n cos(\lambda_n t)$$

Together with (73) this gives us formal expression for $u_3(x_3, t)$:

$$u_3(x_3, t) = \sum_{n=1}^{\infty} X_n(x_3)\left(b_1^n sin(\lambda_n t) + b_2^n cos(\lambda_n t)\right) \tag{80}$$

If we formally take the derivative of (80) with respect to time t we obtain:

$$\frac{du_3(x_3, t)}{dt} = \sum_{n=1}^{\infty} \lambda_n X_n(x_3)\left(b_1^n cos(\lambda_n t) - b_2^n sin(\lambda_n t)\right) \tag{81}$$

In view of initial conditions (54)-(55) from (80) and (81) we formally have

$$\varphi_1(x_3) = \sum_{n=1}^{\infty} X_n(x_3) b_2^n \tag{82}$$

$$\varphi_2(x_3) = \sum_{n=1}^{\infty} \lambda_n X_n(x_3) b_1^n \tag{83}$$

To find expressions for b_1^n and b_2^n let us denote:

$$\Psi_\alpha(x_3) \equiv \frac{L_1}{\rho}(\varphi_{\alpha,3}(x_3)x_3^\kappa)_{,3} \in C([0, L]), \alpha = 1, 2 \tag{84}$$

If we integrate (84) from L to x_3, divide both sides of the obtained equation by x_3 and integrate the result a second time from L to x_3, under the boundary conditions (53) we get:

$$\varphi_\alpha(x_3) = -\rho \int_0^L K(x_3, y)\Psi_\alpha(y)dy, \alpha = 1, 2 \tag{85}$$

where $K(x_3, y)$ is defined by (65).

Because $\Psi_i(\xi) \in C([0, L])$ and $K(x_3, \xi) \in C([0, L] \times [0, L])$ is symmetric (see Lemma 3.1), by Theorem A.1 $\varphi_\alpha(x_3)$ can be represented as the following absolutely and uniformly convergent series on the interval $[0, L]$:

$$\varphi_\alpha(x_3) = \sum_{n=1}^{\infty}\left(\int_0^L \varphi_\alpha(y)X_n(y)dy\right)X_n(x_3), \qquad \alpha = 1, 2 \tag{86}$$

Finally, (86) together with (82) and (83) gives us:

$$b_1^n = \frac{1}{\lambda_n}\int_0^L \varphi_2(y)X_n(y)dy \tag{87}$$

$$b_2^n = \int_0^L \varphi_1(y)X_n(y)dy \tag{88}$$

Absolute and uniform convergence of the series in right-hand side of (80) and (81), as well as of the series for $x^\kappa u_{3,3}(x_3,t)$ and $(x^\kappa u_{3,3}(x_3,t))_{,3}$ in case of homogeneous problem (see eq. (72)) is proved in Section 3.2.

Now, let us consider the case when $f_e(x_3,t) \not\equiv 0$ and $\Phi_3 \not\equiv 0$. Additionally, let us firstly consider the problem when $\varphi_i(x_3)$ given by initial conditions (54)-(55) are equivalently zero on the interval $x_3 \in [0,L]$.

Let $F(x_3,t) \in L_2([0,L])$. Then $F(x_3,t)$ can be represented as:

$$F(x_3,t) = \sum_{n=1}^{\infty} c_n\phi_n$$

where ϕ_n form an orthogonal family in $L_2([0,L])$. Then using Theorem A.6, $F(x_3,t)$ can be represented as a uniformly convergent series:

$$\begin{aligned} F(x_3,t) &= \sum_{n=1}^{\infty}(F(x_3,t),X_n(x_3))X_n(x_3) = \sum_{n=1}^{\infty}\left(\int_0^L F(x_3,t)X_n(x_3)dx_3\right)X_n(x_3) \\ &= \sum_{n=1}^{\infty} F_n(t)X_n(x_3) \end{aligned} \tag{89}$$

where

$$F_n(t) = \int_0^L F(x_3,t)X_n(x_3)dx_3 \tag{90}$$

We look for the solution in the form:

$$u_3(x_3,t) = \sum_{n=1}^{\infty} u_n(x_3,t) \tag{91}$$

where $u_n(x_3,t)$ is a solution of the problem with $F(x_3,t)$ replaced by $X_n(x_3)F_n(t)$. Using the method of separation of variables we can write:

$$u_n(x_3,t) = X_n(x_3)T_{1n}(t) \tag{92}$$

Then from equation (70) we have:

$$\frac{(L_1X_{n,3}(x_3)x_3^\kappa)_{,3}}{X_n(x_3)} = \frac{\rho\ddot{T}_{1n}(t)+F_n(t)}{T_{1n}(t)} = -\lambda_n^2 \tag{93}$$

where $X_n(x_3)$ satisfies (77).

If we solve (93) for $T_{in}(t)$ using the method of variation of parameters, then from (91), (92) and initial-boundary conditions, T_{1n} can be written as:

$$T_{1n} = \frac{\sqrt{\rho}}{\lambda_n} \int_0^t F_n(\tau) sin\left(\frac{\lambda_n}{\sqrt{\rho}}(t-\tau)\right) d\tau \tag{94}$$

Furthermore, from (92) and (94) we get the following series for $u_3(x_3, t)$:

$$u_3(x_3,t) = \sum_{n=1}^{\infty} \frac{\sqrt{\rho}}{\lambda_n} X_n(x_3) \int_0^t \left[\int_0^L F(\xi,\tau) X_n(\xi) d\xi\right] sin\left(\frac{\lambda_n}{\sqrt{\rho}}(t-\tau)\right) d\tau \tag{95}$$

If $F(.,t) \in C([0,L])$ and $F(x_3,.) \in C(t>0) \cap C^1(t>0) \cap C^2(t>0)$, proofs of absolute and uniform convergence of the series in right-hand side of (95), of its first and second order derivatives with respect to time, as well as of the series for $x^\kappa u_{3,3}(x_3,t)$ and $(x^\kappa u_{3,3}(x_3,t))_{,3}$ are given in Section 3.2.2.

Finally, if $\varphi_i(x_3) \not\equiv 0$ then the solution can be expressed as:

$$u_3(x_3,t) = \sum_{n=1}^{\infty} u_n(x_3,t)$$

where

$$u_n(x_3,t) = X_n(x_3)(T_n + T_{1n})$$

$X_n(x_3)T_n$ is here given by (80) and $X_n(x_3)T_{1n}$ is given by (95).

3.2 Absolute Uniform Convergence of the Solution

Remark 3.1. *For simplicity, throughout the following proofs, functions in LHS of inequalities mean the corresponding series.*

3.2.1 Convergence of the solution of homogeneous differential equation

Theorem 3.1. *The series in RHS of (82) and (83) are absolutely and uniformly convergent on $x_3 \in [0,L]$.*

Proof. From (74) and (87) we have:

$$\begin{aligned} b_1^n &= -\frac{L_1}{\lambda_n^3 \rho} \int_0^L \left(X_{n,3}(x_3) x_3^\kappa\right)_{,3} \varphi_2(x_3) dx_3 \\ &= \frac{L_1}{\lambda_n^3 \rho} \int_0^L X_{n,3}(x_3) x_3^\kappa \varphi_{2,3}(x_3) dx_3 \\ &= -\frac{L_1}{\lambda_n^3 \rho} \int_0^L X_n(x_3) \varphi_2(x_3) dx_3 \end{aligned} \tag{96}$$

Analogously:

$$b_2^n = -\frac{L_1}{\lambda_n^2 \rho} \int_0^L X_n(x_3) \varphi_1(x_3) dx_3 \tag{97}$$

As the series given in RHS of (86) is absolutely and uniformly convergent on $[0, L]$, and $K(x_3, \xi) \in C([0, L] \times [0, L])$, from (77) and (97) we have:

$$
\begin{aligned}
|\varphi_1| &\leq \sum_{n=1}^{\infty} |X_n(x_3) b_2^n| = \sum_{n=1}^{\infty} \left| \lambda_n^2 \rho \int_0^L K(x_3, y) X_n(y) b_2^n dy \right| \\
&\leq |L_1| \sum_{n=1}^{\infty} \left| \int_0^L K(x_3, y) \left[\int_0^L X_n(\xi) \varphi_1(\xi) X_n(y) d\xi \right] dy \right| \\
&\leq |L_1| \int_0^L |K(x_3, y)| \sum_{n=1}^{\infty} \left| \int_0^L X_n(\xi) \varphi_1(\xi) X_n(y) d\xi \right| dy \\
&\leq |L_1| \int_0^L |K(x_3, y)| M(y) dy \leq |L_1| M \int_0^L |K(x_3, y)| dy < \infty
\end{aligned}
$$

where

$$
M(y) = \sum_{n=1}^{\infty} \left| \int_0^L X_n(\xi) \varphi_1(\xi) X_n(y) d\xi \right|
$$

RHS of last expression is absolutely and uniformly convergent on $[0, L]$ and $M = \max_{0 \leq y \leq L} M(y)$. Analogously using (96) we have:

$$
|\varphi_2| \leq \sum_{n=1}^{\infty} |\lambda_n X_n(x_3) b_1^n| < \infty
$$

□

Theorem 3.2. *The series in RHS of (80), as well as its first and second order derivatives with respect to time t is absolutely and uniformly convergent on $x_3 \in [0, L]$.*

Proof. From (80), using results from Theorem 3.1, we have:

$$
\begin{aligned}
|u_3(x_3, t)| &\leq \sum_{n=1}^{\infty} |X_n(x_3)| |b_1^n| \, |\sin(\lambda_n t)| + \sum_{n=1}^{\infty} |X_n(x_3)| |b_2^n| \, |\cos(\lambda_n t)| \\
&\leq \frac{1}{\lambda_0} \sum_{n=1}^{\infty} |\lambda_n X_n(x_3) b_1^n| + \sum_{n=1}^{\infty} |X_n(x_3) b_2^n| < \infty
\end{aligned}
$$

From (81) and absolute uniform convergence of the series in RHS of (86) we have:

$$
\begin{aligned}
|\dot{u}_3(x_3,t)| &\leq \sum_{n=1}^{\infty} \lambda_n |X_n(x_3)||b_1^n cos(\lambda_n t)| \\
&+ \sum_{n=1}^{\infty} \lambda_n |X_n(x_3)||b_2^n sin(\lambda_n t)| \\
&\leq \frac{L_1}{\lambda_0^2 \rho} \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_2(\xi) d\xi \right| \\
&+ \frac{L_1}{\lambda_0 \rho} \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_1(\xi) d\xi \right| \\
&\leq \frac{L_1}{\lambda_0^2 \rho} M_2(x_3) + \frac{L_1}{\lambda_0 \rho} M_1(x_3) < \infty
\end{aligned}
$$

where $\lambda_0^2 = \underset{n}{min} \lambda_n^2$ and

$$
M_\alpha(x_3) = \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_\alpha(\xi) d\xi \right| \tag{98}
$$

Analogously,

$$
\begin{aligned}
|\ddot{u}_3(x_3,t)| &\leq \frac{L_1}{\lambda_0 \rho} \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_2(\xi) d\xi \right| \\
&+ \frac{L_1}{\rho} \sum_{n=1}^{\infty} |X_n(x_3)| \left| \int_0^L X_n(\xi)\varphi_1(\xi) d\xi \right| < \infty
\end{aligned}
$$

□

Remark 3.2. *As it was proved in Lemma 3.2, eigenvalues λ_n^2 are infinite in number. Furthermore, it was shown that the kernel $K(x_3, y)$ given by (65) is symmetric (see Lemma (3.1)). In ([22]) it is shown, that in case of symmetric kernel, to each eigenvalue belongs normalized orthogonal system of eigenfunctions and there exists at least one eigenvalue. Additionally, if they are infinite in number, they form a denumerable set and they may be arranged in the order of magnitude of their absolute values:*

$$
|\lambda_1^2| \leq |\lambda_2^2| \leq ... \leq |\lambda_n^2| \leq ...
$$

Consequently, we can chose λ_0^2 such that $\lambda_0^2 = \underset{n}{min} \lambda_n^2$.

Theorem 3.3. *The corresponding series of $x_3^\kappa u_{3,3}(x_3, t)$ is absolutely and uniformly convergent on $x_3 \in [0, L]$.*

Proof. From (74) using boundary conditions (53) we will get:

$$
X_{n,3}(x_3) = -\frac{1}{x_3^\kappa} \frac{\rho \lambda_n^2}{L_1} \int_0^L K_1(x_3, \xi) X_n(\xi) d\xi
$$

where

$$K_1(x_3,\xi)=\begin{cases}\frac{\xi^{1-\kappa}}{L^{1-\kappa}} & 0\leq\xi<x_3\\ \frac{\xi^{1-\kappa}}{L^{1-\kappa}}-1 & x_3\leq\xi\leq L\end{cases}$$

Together with (80) and (96)-(97):

$$\begin{aligned}|x_3^\kappa u_{3,3}\ (x_3,t)| &= \left|x_3^\kappa\sum_{n=1}^{\infty}X_{n,3}(x_3)\left(b_1^n sin\left(\lambda_n t\right)+b_2^n cos\left(\lambda_n t\right)\right)\right|\\ &=\left|\frac{\rho}{L_1}\sum_{n=1}^{\infty}\lambda_n^2\int_0^L K_1(\xi)X_n(\xi)\left(b_1^n sin\left(\lambda_n t\right)+b_2^n cos\left(\lambda_n t\right)\right)d\xi\right|\\ &\leq M_k\left[\frac{1}{\lambda_0}\sum_{n=1}^{\infty}\int_0^L\left|X_n(\xi)\int_0^L X_n(\eta)\varphi_2(\eta)d\eta\right|d\xi\right.\\ &\left.+\sum_{n=1}^{\infty}\int_0^L\left|X_n(\xi)\int_0^L X_n(\eta)\varphi_1(\eta)d\eta\right|d\xi\right]\\ &\leq M_k\left[\frac{M_2}{\lambda_0}+M_1\right]<\infty\end{aligned}$$

where $M_k=\underset{\xi}{max}K_1(\xi)$, $\lambda_0^2=\underset{n}{min}\lambda_n^2$ and M_α, $\alpha=1,2$ is defined by (98). □

Theorem 3.4. *The corresponding series of $(x_3^\kappa u_{3,3}(x_3,t))_{,3}$ is absolutely and uniformly convergent on $x_3\in(0,L]$.*

Proof. Using the result of Theorem 3.3 and proceeding in the same way, we get:

$$\begin{aligned}\left|\left(x_3^\kappa u_{3,3}(x_3,t)\right)_{,3}\right| &= \left|\kappa x_3^{\kappa-1}u_{3,3}+x_3^\kappa u_{3,33}\right|\\ &\leq\frac{2\kappa\rho}{x_3L_1}\left|\sum_{n=1}^{\infty}\lambda_n^2\int_0^L K_1(\xi)X_n(\xi)\left(b_1^n sin\left(\lambda_n t\right)\right.\right.\\ &\left.\left.+b_2^n cos\left(\lambda_n t\right)\right)d\xi\right|\leq\frac{2\kappa C^*}{x_3}\end{aligned}$$

where C^* is constant such that $|x^\kappa u_{3,3}(x_3,t)|\leq C^*$ (see Theorem 3.3). □

3.2.2 Convergence of the solution of non-homogeneous differential equation

Remark 3.3. *Note, that from (77) and (89) $F(x_3,t)$ can be written in the form:*

$$F(x_3,t)=\int_0^L K(x_3,y)g(x_3,y,t)dy$$

where $g(x_3,y,t)\in C([0,L],[0,L],t>0)$. This result will be used along with Theorem A.3 in the following theorems.

Theorem 3.5. *If $F(x_2,t)\in C([0,L],t>0)$, the series in RHS of (95) is absolutely and uniformly convergent on $x_3\in[0,L]$.*

Proof.
$$|u_3(x_3,t)| \leq \left| \frac{1}{\rho} \int_0^t \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \int_0^L F(\xi,\tau) X_n(x_3)(\xi) d\xi \right) X_n(x_3) \right|$$
If conditions of the theorem holds for $F(x_3,t)$ then from Theorem A.3 and Remark 3.3
$$\sum_{n=1}^{\infty} \left(\int_0^L F(\xi,\tau) X_n(x_3)(\xi) d\xi \right) X_n(x_3)$$
is absolutely and uniformly convergent on $[0,L]$, thus
$$\sum_{n=1}^{\infty} \left(\int_0^L F(\xi,\tau) X_n(x_3)(\xi) d\xi \right) X_n(x_3) \leq c(\tau)$$
and
$$|u_3(x_3,t)| \leq \left| \frac{1}{\rho\lambda_0} \int_0^t c(\tau) d\tau \right| < \infty$$
where $\lambda_0^2 = \underset{n}{min} \lambda_n^2$. □

Theorem 3.6. *If $F(.,t) \in C([0,L])$ and $F(x_3,.) \in C(t>0) \cap C^1(t>0) \cap C^2(t>0)$ then first and second order derivatives of the series given in RHS of (95) with respect to time is absolutely and uniformly convergent on $x_3 \in [0,L]$.*

Proof. Similarly to the proof of Theorem (3.5) we have:
$$\begin{aligned} |\dot{u_3}(x_3,t)| \leq & \left| \frac{1}{\rho} \int_0^t \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} \int_0^L \dot{F}(\xi,\tau) X_n(x_3)(\xi) d\xi \right) X_n(x_3) \right| \\ & + \left| \frac{1}{\rho} \int_0^t \sum_{n=1}^{\infty} \left(\int_0^L F(\xi,\tau) X_n(x_3)(\xi) d\xi \right) X_n(x_3) \right| < \infty \end{aligned}$$
□

Theorem 3.7. *If $F(x_2,t) \in C([0,L], t>0)$, the corresponding series of $x_3^{\kappa} u_3(x_3,t)$ is absolutely and uniformly convergent on $x_3 \in [0,L]$.*

Proof. Similarly to the proof of Theorem 3.5:
$$|x_3^{\kappa} u_3(x_3,t)| \leq \left| \frac{x_3^{\kappa}}{\rho\lambda_0} \int_0^t c(\tau) d\tau \right| < \infty$$
□

Theorem 3.8. *If $F(x_2,t) \in C([0,L], t>0)$, the corresponding series of $(x_3^{\kappa} u_3(x_3,t))_{,3}$ is absolutely and uniformly convergent on $x_3 \in (0,L]$.*

Proof. Similarly to the proofs of Theorem 3.5 and Theorem 3.7:

$$|(x_3^\kappa u_3(x_3,t))_{,3}| \le \left|\frac{x_3^\kappa}{\rho\lambda_0}\int_0^t c(\tau)d\tau\right| + \left|\frac{\kappa x_3^{\kappa-1}}{\rho\lambda_0}\int_0^t c(\tau)d\tau\right| < \infty$$

□

4 Analytical Solutions with Matlab

In the following section numerical results are shown for the solutions of the problems discussed in Section 2. Coefficients of mixture of $CoFe_2O_4$ (PM) and $BaTiO_3$ (PE) with volume fraction $\frac{c_{PE}}{c_{PM}} = .25$ was taken during the calculations [32]:

$$\begin{pmatrix} E_{1333} & E_{2333} & E_{3333} \\ p_{313} & p_{323} & p_{333} \\ q_{313} & q_{323} & q_{333} \\ \varsigma_{33} & \tilde{a}_{33} & \xi_{33} \end{pmatrix} = \begin{pmatrix} E^0_{1333} & E^0_{2333} & E^0_{3333} \\ p^0_{313} & p^0_{323} & p^0_{333} \\ q^0_{313} & q^0_{323} & q^0_{333} \\ \varsigma^0_{33} & \tilde{a}^0_{33} & \xi^0_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 243\times 10^9 \\ 0 & 0 & 4.65 \\ 0 & 0 & 12.6 \\ 3.22\times 10^{-9} & 0 & 1.2\times 10^{-4} \end{pmatrix}$$

and $\rho = 5.43 \times 10^3$.

The thickness L of the plate was taken to equal to 10.

Remark 4.1. *Units of all constants are in the SI system.*

Remark 4.2. *MATLAB codes for the solutions of the problems discussed in this section are provided in Appendix B.*

4.1 Constitutive Coefficients are Constants

4.1.1 Static Problem

In this section the plots of the solutions of the system of equations (15), (16), (18) are provided. The solution for u_3, η and χ were obtained from equations (20), (21) and (22) using MATLAB function 'dsolve'. Two cases are considered: Figure 4.1 shows plots of solutions when boundary conditions (19) were taken to be homogeneous and known functions (charge density f_e and volume force along x_3 axis Φ_3) does not depend on spatial variable x_3; Figure 4.2 shows plots for non-homogeneous boundary conditions case when given volume force depends on spatial variable x_3 ($\Phi_3 = 10^3 x_3$).

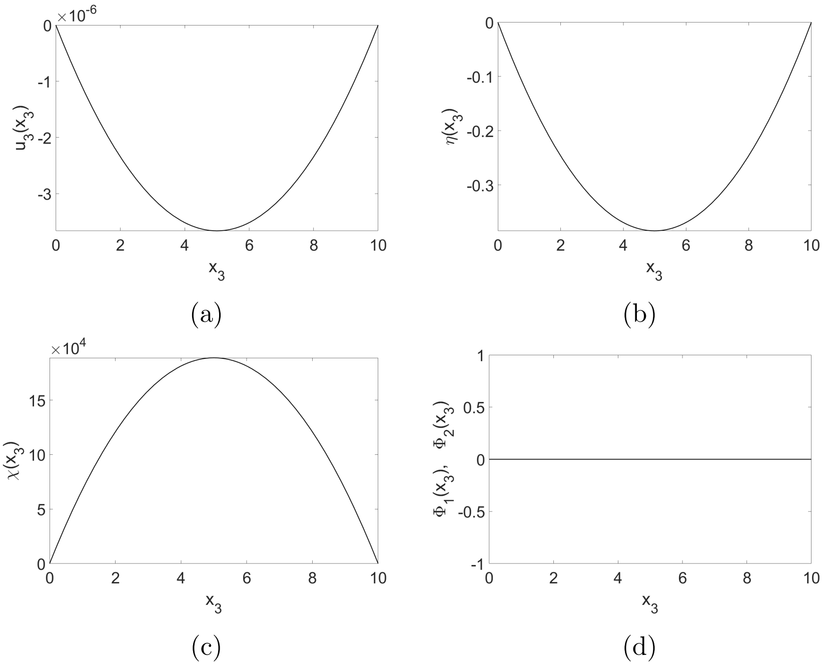

Figure 4.1: Displacement (a), magnetic potential (b), electric potential (c) and volume force components along x_1 and x_2 axes (d) as functions of x_3. $f_e(x_3) = 5 \times 10^{-5}$, $\Phi_3(x_3) = 10^3$; homogeneous boundary conditions.

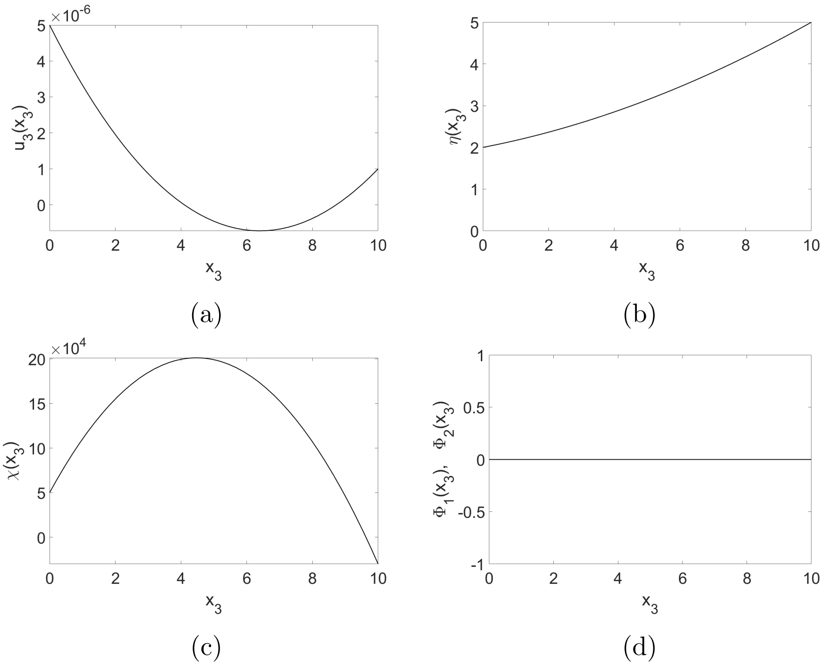

Figure 4.2: Displacement (a), magnetic potential (b), electric potential (c) and volume force components along x_1 and x_2 axes (d) as functions of x_3. $f_e(x_3) = 10^{-5}$, $\Phi_3(x_3) = 10^3 x_3$; Boundary conditions: $a_0 = 5 \times 10^{-6}$, $a_1 = 10^{-6}$, $b_0 = 5 \times 10^4$, $b_1 = -3 \times 10^4$, $d_0 = 2$, $d_1 = 5$.

4.1.2 Dynamic Problem

In this section plots of the solutions of the system of equations (14)-(16) are provided. The solution for u_3, η and χ were obtained from equations (38), (21) and (22) using MATLAB function 'dsolve'. Three cases are considered: Figure 4.3 shows the plots of solutions for non-homogeneous boundary conditions (19) when known functions (charge density f_e and volume force along x_3 axis Φ_3) does not depend on spatial variable x_3; Figure 4.4 shows the plots of the solutions for homogeneous boundary conditions when volume force along x_3 has the following form: $\Phi_3 = 10^5 x_3$; Figure 4.5 shows the plots of the solutions for homogeneous boundary conditions when volume force along x_3 axis has the following form: $\Phi_3 = 10^5 cos(x_3)$.

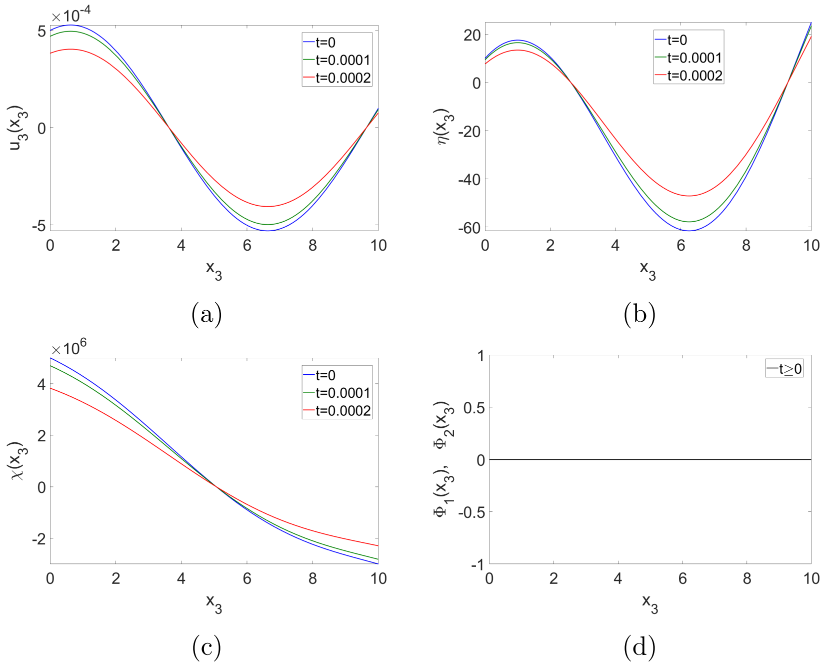

Figure 4.3: Displacement (a), magnetic potential (b), electric potential (c) and volume force components along x_1 and x_2 axes (d) as functions of x_3. $f_e(x_3) = 7 \times 10^{-8}$, $\Phi_3(x_3) = 10^5$; Boundary conditions: $a^* = 5 \times 10^{-4}$, $a^* = 10^{-4}$, $a_0 = 5 \times 10^6$, $a_L = -3 \times 10^6$, $b_0 = 10$, $b_L = 25$.

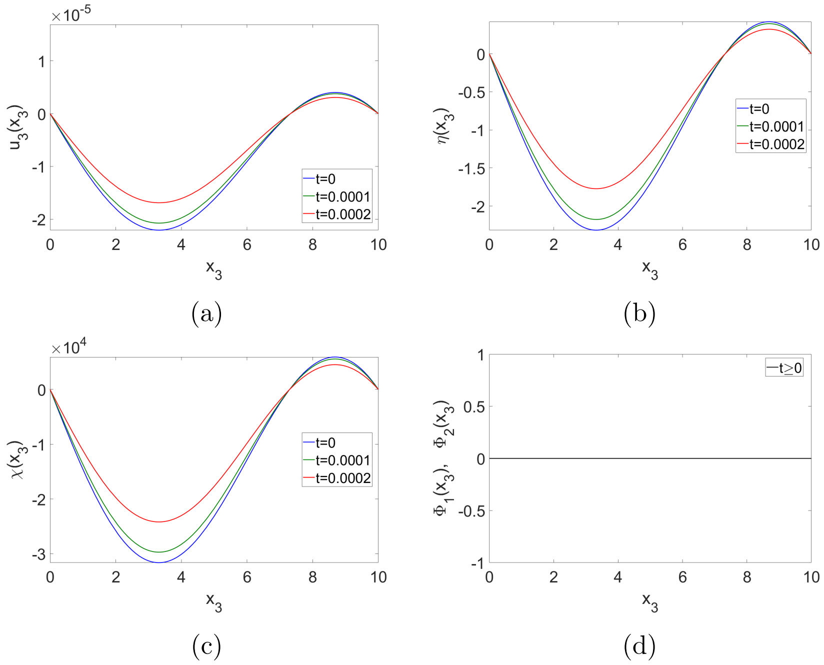

Figure 4.4: Displacement (a), magnetic potential (b), electric potential (c) and volume force components along x_1 and x_2 axes (d) as functions of x_3. $f_e(x_3) = 7 \times 10^{-8}$, $\Phi_3(x_3) = 10^5 x_3$; homogeneous boundary conditions.

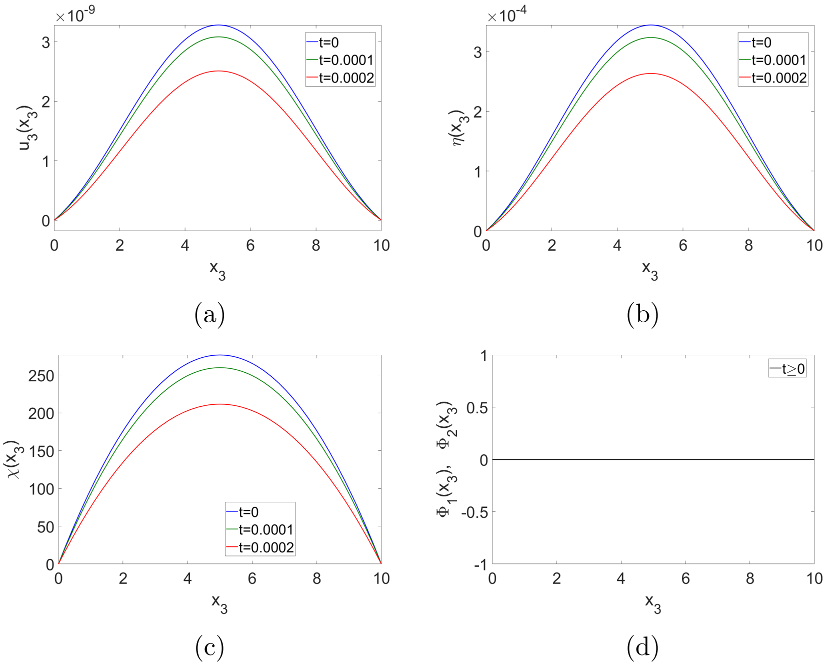

Figure 4.5: Displacement (a), magnetic potential (b), electric potential (c) and volume force components along x_1 and x_2 axes (d) as functions of x_3. $f_e(x_3) = 7 \times 10^{-8}$, $\Phi_3(x_3) = cos(x_3)$; homogeneous boundary conditions.

Remark 4.3. *As it can be seen on the figures, for considered material* Φ_α, $\alpha = 1, 2$ *is zero on the interval* $x_3 \in [0, L]$ *in both static and dynamic cases. This means, that no additional forces are needed along* x_1 *and* x_2 *axis in order to limit the deformation only along* x_3 *axes.*

Conclusions

Static and dynamic problems of non-homogeneous piezoelectric elastic beam was studied in case when constitutive coefficients are constants and when they are power functions of spatial variable x_3, i.e. equal to $const. \times x_3^{\kappa}$ for some $\kappa = const. \in (0, 1)$. When constitutive coefficients are constants, the solutions were written in analytic form as functions of time t and/or spatial variable x_3. When constitutive coefficients are power functions of x_3 the solutions were written in absolute uniform convergent series and it was proved, that when constitutive coefficients are power functions of x_3, the solution is oscillatory iff L_1 constant defined by (66) is positive.

In the case when constitutive coefficients were constants several problems were solved for different boundary conditions and known functions (charge density f_e and volume force along spatial variable x_3 - Φ_3). Solutions of the problems were plotted using MATLAB.

References

[1] Eringen A.C. *Mechanics of continua.* Krieger, 01 1980.

[2] D.N. Astrov. Sur l'électricité polaire dans les cristaux hémièdres ‘ faces inclinées. *Journal of Experimental and Theoretical Physics*, 1960.

[3] T. Buchukuri, O. Chkadua, Duduchava R., and Natroshvili D. Interface crack problems for metallic-piezoelectric composite structures. *Memoirs on Differential Equations and Mathematical Physics*, 55, 01 2012.

[4] Natalia Chinchaladze. On some analytic methods for calculating of cusped prismatic shells. *PAMM*, 14, 12 2014.

[5] J. Curie and Curie. P. Développement, par pression, de l'électricité polaire dans les cristaux hémièdres faces ‘ inclinées. *C R Acad Sci Gen 91:294–295*, 1880.

[6] J. Curie and Curie. P. Sur l'électricité polaire dans les cristaux hémièdres ‘ faces inclinées. *C R Acad Sci Gen 91:383–386*, 1880.

[7] Natroshvili D. *Mathematical Problems of Thermo-Electro-Magneto-Elasticity*, volume 12 of *Lecture Notes of TICMI.* Tbilisi University Press, 2011.

[8] Moon F. and Graneau P. Magneto-solid mechanics. *Physics Today*, 38:79–, 01 1985.

[9] V. J. Folen, G. T. Rado, and E. W. Stalder. Anisotropy of the magnetoelectric effect in cr_2o_3. *Phys. Rev. Lett.*, 6:607–608, Jun 1961.

[10] Amelchenko A. G., Bardin V. A., VasiFev V. A., Krevchick V. D., Chernov P. S., and Shcherbakov M. A. Piezo actuators and piezo motors for driving systems. In *2016 Dynamics of Systems, Mechanisms and Machines (Dynamics)*, pages 1–4, 2016.

[11] Jaiani G. Differential hierarchical models for elastic prismatic shells with microtemperatures. *ZAMM Journal of applied mathematics and mechanics: Zeitschrift für angewandte Mathematik und Mechanik*, 95, 09 2013.

[12] Jaiani G. Hierarchical models for viscoelastic kelvin-voigt prismatic shells with voids. *Bulletin of TICMI*, 21:33–44, 01 2017.

[13] Jaiani G. *Piezoelectric Viscoelastic Kelvin-Voigt Cusped Prismatic Shells*, volume 19 of *Lecture Notes of TICMI.* Tbilisi University Press, 2018.

[14] Jaiani G. and Bitsadze L. On basic problems for elastic prismatic shells with microtemperatures. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 96, 12 2015.

[15] Jaiani G. and Bitsadze L. Basic problems of thermoelasticity with microtemperatures for the half-space. *Journal of Thermal Stresses*, 41:1–14, 07 2018.

[16] Vekua I. On a way of calculating of prismatic shells. *Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences*, pages 21:191–259, 1955. (Russian).

[17] Vekua I. The theory of thin shallow shells of variable thickness. *Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences*, pages 30:5–103, 1965. (Russian).

[18] R.P. Kanwal. *Linear Integral Equations: Theory & Technique*. Modern Birkhäuser Classics. Springer New York, 2012.

[19] P. Kythe and P. Puri. *Computational Methods for Linear Integral Equations*. Springer Science+Business Media, LLC, 01 2002.

[20] L.D. Landau and E.M. Lifshitz. *Electrodynamics of continuous media; 2nd ed.* Course of theoretical physics. Butterworth, Oxford, 1984.

[21] G. Lippmann. Principe de la conservation de l'électricité, ou second principe de la théorie des phénomènes électriques. *J. Phys. Theor. Appl.*, 10(1):381–394, 1881.

[22] W.V. Lovitt. *Linear Integral Equations*. Dover books on mathematics. Dover Publications, 1950.

[23] Taha M., Walia S., Ahmed T., Headland D., Withayachumnankul W., Sriram S., and Bhaskaran M. Insulator-metal transition in substrate-independent vo2 thin film for phase-change devices. *Scientific Reports*, 7(1), 2017.

[24] Trindade M. Applications of piezoelectric sensors and actuators for active and passive vibration control. 05 2008.

[25] Chinchaladze N. *On Some Nonclassical Problems for Differential Equations and Their Applications to the Theory of Cusped Prismatic Shells*, volume 9 of *Lecture Notes of TICMI*. Tbilisi University Press, 2008.

[26] Chinchaladze N. On one problem of a cusped elastic prismatic shells in case of the third model of vekua s hierarchical model. *Hacettepe Journal of Mathematics and Statistics*, 5, 12 2016.

[27] Chinchaladze N. and Tutberidze M. On some bending problems of prismatic shell with the thickness vanishing at infinity. *Journal of Mathematics and System Science*, 7, 03 2017.

[28] Chinchaladze N. and Gilbert R. Harmonic vibration of prismatic shells in zero approximation of vekua's hierarchical models. *Applicable Analysis*, 92:2275–2287, 11 2013.

[29] Mittal N., Ansari F., Gowda V. K., Brouzet Ch., Chen P., Larsson P., Roth S., Lundell F., Wågberg L., Kotov N., and Söderberg D. Multiscale control of nanocellulose assembly: Transferring remarkable nanoscale fibril mechanics to macroscale fibers. *ACS Nano*, 12, 05 2018.

[30] Cugat O., Delamare J., and Reyne G. Magnetic micro-actuators systems (magmas). In *2003 IEEE International Magnetics Conference (INTERMAG)*, pages GB–04, 2003.

[31] Dineva P., Gross D., Müller R., and Rangelov T. *Dynamic Fracture of Piezoelectric Materials*, volume 212. 01 2014.

[32] E. Pan and W. Chen. *Static Green's Functions in Anisotropic Media*. Cambridge University Press, 2015.

[33] Toupin R.A. A dynamical theory of elastic dielectrics. *International Journal of Engineering Science*, 1:101–126, 03 1963.

[34] Lang S. Guide to the literature of piezoelectricity and pyroelectricity. 28. *Ferroelectrics*, 361:130–216, 12 2007.

[35] Buchukuri T., Chkadua O., and Natroshvili D. Mathematical problems of generalized thermo-electro-magneto-elasticity theory. *Memoirs on Differential Equations and Mathematical Physics*, 68:1–166, 01 2016.

[36] Buchukuri T., Chkadua O., and Natroshvili D. Mixed boundary value problems of pseudo-oscillations of generalized thermo-electro-magneto-elasticity theory for solids with interior cracks. *Transactions of A. Razmadze Mathematical Institute*, 170, 09 2016.

[37] Nowacki W. *Efecty electromagnetyczne w stalych cialach odksztalcalnych. Warszawa*. Panstwowe Widawnictwo Naukowe, 1983. [Russian translation: Electromagnetic effects in solids, Moscow, Mir, 1986].

[38] Y. Wang and X. Xia. Magnetoelectric coupling and interface effects of multiferroic composites under stress-prescribed boundary condition. *Reviews on Advanced Materials Science*, 48:78–90, 01 2017.

[39] Boyce W.E. and DiPrima R.C. *Elementary Differential Equations and Boundary Value Problems*. 2001.

Appendix A Supplementary (Hilbert-Schmidt) Theorems

The theorems given in this section as well as their proofs can be found in (Lovitt. *Linear Integral Equations.* 2005) and (Kanwal R.P., Linear Integral Equations, Second Edition, 1997).

Theorem A.1. *If $u(x_3)$ has the form*

$$u(x_3) = \lambda \int_0^L R(x_3, \xi) f(\xi) d\xi$$

with $f(x_3)$ piece-wise continuous on $[0, L]$, and a symmetric kernel $R(x_3, \xi) \in C([0, L] \times [0, L])$, then

$$u(x_3) = \sum_{n=1}^{\infty} (u, Y_n) Y_n(x_3) \tag{99}$$

where

$$(u, Y_n) := \int_0^L u(x_3) Y_n(x_3) dx_3$$

Y_n is an eigenfunction of $R(x_3, \xi)$, and the series on the right-hand side of (99) is convergent absolutely and uniformly on $[0, L]$.

Theorem A.2. *If the number of λ_n eigenvalues of the symmetric kernel is finite then*

$$R(x_3, \xi) = \sum_{n=1}^{N} \frac{Y_n(x_3) Y_n(\xi)}{\lambda_n}$$

Theorem A.3. *If $f(x_3) \in C([0, L])$, then*

$$\int_0^L R(x_3, \xi) f(\xi) d\xi = \sum_{n=1}^{\infty} \frac{(f, Y_n)}{\lambda_n} Y_n,$$

and the series is convergent absolutely and uniformly, here $R(x_3, \xi)$ is a symmetric kernel with respect to x_3 and ξ; Y_n are eigenfunctions of R corresponding to λ_n eigenvalues.

Theorem A.4. *To every real symmetric kernel $K(x, t)$, $x, t \in [a, b]$ there belongs a complete normalized orthogonal system of eigenfunctions $\psi_r(x)$, with the following properties:*

1) $\psi_r(x)$ is a eigenfunction belonging to λ_r

$$\psi_r(x) = \lambda_r \int_a^b K(x, t) \psi_r(t) dt$$

2) $\int_a^b \psi_r^2(x) dx = 1$

3) $\int_a^b \psi_r(x) \psi_s(x) dx = 0, \qquad (r \neq s)$

4) $\psi_r(x)$ is real.

5) Every eigenfunction $\varphi(x)$ is expressible in the form

$$\varphi(x) = c_{r_1}\psi_{r_1}(x) + ... + c_{r_m}\psi_{r_m}(x)$$

Theorem A.5. *If $K(x,t)$ is real and symmetric, continuous, and $\not\equiv 0$, then all of the eigenvalues are real.*

Theorem A.6. *If $f(x)$ is a continuous function and its expressible in the form*

$$f(x) = \sum_{n=1}^{\infty} c_n\psi_n(x)$$

where infinite sum in RHS is uniformly convergent on $[a,b]$, then the coefficients c_n are given by

$$c_n = \int_a^b f(x)\psi_n(x)dx \equiv (f\psi_n)$$

that is

$$f(x) = \sum_{n=1}^{\infty}(f\psi_n)\psi_n(x)$$

where ψ_n belongs to a complete normalized orthogonal system of eigenfunctions of symmetric kernel.

Appendix B Matlab Codes for Section 4

B.1 Constitutive Coefficients are Constants

B.1.1 Static Problem

```
syms f_e(x_3) Phi_3(x_3) u_3(x_3) eta(x_3) chi(x_3) Phi_alpha(x_3)
%% Defining constants and given functions
L = 10; % Thickness of the plate
rho = 5.43*10^3; % Density of the material

f_e(x_3) = 5*10^(-5); % Charge density function
% Phi_3(x_3) = 10^3; % Volume force along x_3 axis (Figure 6.1)
Phi_3(x_3) = 10^3*x_3; % Volume force along x_3 axis (Figure 6.2)


% Constitutive Coefficients
E_i333 = [0 0 243*10^9];
p_3i3 = [0 0 4.65];
q_3i3 = [0 0 12.6];
sigma_33 = 3.22*10^(-9);
a_33 = 0;
xi_33 = 1.2*10^(-4);

% Constants constructed from constitutive coefficients
L_1i = E_i333 + p_3i3*(xi_33*p_3i3(3)-a_33*q_3i3(3)) - ...
    q_3i3*(a_33*p_3i3(3)-sigma_33*q_3i3(3))/(xi_33*sigma_33-(a_33)^2);
L_2i = (p_3i3*xi_33-q_3i3*a_33)/(xi_33*sigma_33-(a_33)^2);
L_3 = xi_33/(xi_33*sigma_33-(a_33)^2);
L_4 = (p_3i3(3)*a_33 - sigma_33*q_3i3(3))/((a_33)^2-xi_33*sigma_33);
L_5 = a_33/((a_33)^2-xi_33*sigma_33);

% Non-homogeneous Boundary Conditions
a_0 = 5*10^(-6);
a_1 = 10^(-6);
b_0 = 5*10^4;
b_1 = -3*10^4;
d_0 = 2;
d_1 = 5;

%% Solving differential equations
D2uDx2 = diff(u_3,2);
D2etaDx2 = diff(eta,2);
D2chiDx2 = diff(chi,2);
```

```
eq1 = L_1i(3)*D2uDx2 - L_2i(3)*f_e(x_3) + Phi_3(x_3) == 0;
bc1='u_3(0)=a_0,u_3(L)=a_1';
S1=dsolve(eq1,bc1,x_3);
u_3(x_3)=eval(vectorize(S1));

eq2 = D2etaDx2 == L_4*diff(u_3,2) - L_5*f_e(x_3);
bc2='eta(0)=d_0,eta(L)=d_1';
S2=dsolve(eq2,bc2,x_3);
eta(x_3)=eval(vectorize(S2));

eq3 = D2chiDx2 == L_2i(3)*diff(u_3,2) - L_3*f_e(x_3);
bc3='chi(0)=b_0,chi(L)=b_1';
S3=dsolve(eq3,bc3,x_3);
chi(x_3)=eval(vectorize(S3));

Phi_alpha(x_3)=(L_2i(1:2)-(L_1i(1:2)/L_1i(3))*L_2i(3))*f_e(x_3)+...
    (L_1i(1:2)/L_1i(3))*Phi_3(x_3);

%% Plotting solutions
figure(1)
fplot(u_3,[0 L],'k','linewidth',2)
xlabel('x_3')
ylabel('u_3(x_3)')

figure(2)
fplot(eta,[0 L],'k','linewidth',2)
xlabel('x_3')
ylabel('\eta(x_3)')

figure(3)
fplot(chi,[0 L],'k','linewidth',2)
xlabel('x_3')
ylabel('\chi(x_3)')

figure(4)
fplot(Phi_alpha(1),[0 L],'k','linewidth',2)
xlabel('x_3')
ylabel('\Phi_1(x_3),   \Phi_2(x_3)')
```

B.1.2 Dynamic Problem

```
syms f_e(x_3) Phi(x_3) u_3(x_3) eta(x_3) chi(x_3) Phi_alpha(x_3)...
    g(x_3)
%% Defining constants and given functions
L = 10; % Thickness of the plate
rho = 5.43*10^3; % Density of the material
omega = 3500; % Frequency of oscillation

f_e(x_3) = 7*10^(-8); % Charge density function (see eq. (30))
% Phi(x_3) = 10^5;  % Volume force along x_3 axis (Figure 6.3)
% Phi(x_3) = 10^5*x_3; % Volume force along x_3 axis (Figure 6.4)
Phi(x_3) = cos(x_3);  % Volume force along x_3 axis (Figure 6.5)

% Constitutive Coefficients
E_i333 = [0 0 243*10^9];
p_3i3 = [0 0 4.65];
q_3i3 = [0 0 12.6];
sigma_33 = 3.22*10^(-9);
a_33 = 0;
xi_33 = 1.2*10^(-4);

% Constants constructed from constitutive coefficients
L_1i = E_i333 + p_3i3*(xi_33*p_3i3(3)-a_33*q_3i3(3)) - ...
   q_3i3*(a_33*p_3i3(3)-sigma_33*q_3i3(3))/(xi_33*sigma_33-(a_33)^2);
L_2i = (p_3i3*xi_33-q_3i3*a_33)/(xi_33*sigma_33-(a_33)^2);
L_3 = xi_33/(xi_33*sigma_33-(a_33)^2);
L_4 = (p_3i3(3)*a_33 - sigma_33*q_3i3(3))/((a_33)^2-xi_33*sigma_33);
L_5 = a_33/((a_33)^2-xi_33*sigma_33);

omega_0 = sqrt(rho*omega^2/L_1i(3));
% RHS of differential equation
g(x_3) = L_2i(3)*f_e(x_3)/L_1i(3) - Phi(x_3)/L_1i(3);

% Non-homogeneous Boundary Conditions
a = 0; %5*10^(-4);
b = 0; %10^(-4);
a_0 = 0; %5*10^6;
a_L = 0; %-3*10^6;
b_0 = 0; %10;
b_L = 0; %25;

%% Solving differential equations
```

```
D2uDx2 = diff(u_3,2);
eq1 = D2uDx2 == -omega_0^2*u_3 + g(x_3);
bc1='u_3(0)=a,u_3(L)=b';
S1=dsolve(eq1,bc1,x_3);
u_3(x_3) = eval(vectorize(S1));

D2etaDx2 = diff(eta,2);
eq2 = D2etaDx2 == L_4*diff(u_3,2) - L_5*f_e(x_3);
bc2='eta(0)=b_0,eta(L)=b_L';
S2=dsolve(eq2,bc2,x_3);
eta(x_3)=eval(vectorize(S2));

D2chiDx2 = diff(chi,2);
eq3 = D2chiDx2 == L_2i(3)*diff(u_3,2) - L_3*f_e(x_3);
bc3='chi(0)=a_0,chi(L)=a_L';
S3=dsolve(eq3,bc3,x_3);
chi(x_3)=eval(vectorize(S3));

Phi_alpha = -L_1i(1:2)*u_3(x_3) + L_2i(1:2)*f_e(x_3);

%% Plotting solutions (Animation)
%Initialize plots
filename = '(Fig. 6.5)din_constCoeff_nonconstFun2_homogen.gif';
N = 100;
x = 0:1/N:L;
u3Data = u_3(x);
etaData = eta(x);
chiData = chi(x);

h = figure(1);
axis tight manual
subplot(221)
u3plot = plot(x,zeros(size(x)),'k','linewidth',2);
ylim([-4*10^(-9) 4*10^(-9)]) % Scaled for specific solution
xlabel('x_3')
ylabel('u_3(x_3)')

subplot(222)
etaplot = plot(x,zeros(size(x)),'k','linewidth',2);
ylim([-4*10^(-4) 4*10^(-4)]) % Scaled for specific solution
xlabel('x_3')
ylabel('\eta(x_3)')
```

```
subplot(223)
chiPlot = plot(x,zeros(size(x)),'k','linewidth',2);
ylim([-300 300]) % Scaled for specific solution
xlabel('x_3')
ylabel('\chi(x_3)')

subplot(224)
fplot(Phi_alpha(1),[0 L],'k','linewidth',2)
hold on
fplot(Phi_alpha(2),[0 L],'k','linewidth',2)
hold off
xlabel('x_3')
ylabel('\Phi_1(x_3),   \Phi_2(x_3)')

% Making Animation
for t=0:0.00005:0.0018
    u3plot.YData = u3Data*cos(omega*t);
    etaplot.YData = etaData*cos(omega*t);
    chiPlot.YData = chiData*cos(omega*t);
    suptitle(strcat('t = ',num2str(t)));
    drawnow();
    % Capture the plot as an image and save to .gif
    frame = getframe(h);
    im = frame2im(frame);
    [imind,cm] = rgb2ind(im,256);
    if t==0
        imwrite(imind,cm,filename,'gif', 'Loopcount',inf);
    else
        imwrite(imind,cm,filename,'gif','WriteMode','append');
    end
end
```

Appendix C Additional Remarks

C.1 On the Definition of Eigenvalues and Eigenfunctions

If we write the homogeneous Fredholm equation as

$$\int_a^b K(s,t)g(t)dt = \mu g(s), \quad \mu = 1/\lambda$$

we have the classical eigenvalue or characteristic value problem. In the literature on integral equations some authors use the notation λ for eigenvalues only, whereas others use μ for eigenvalues or characteristic values such that $\lambda\mu = 1$.

Since we study integral equation of the form

$$g(s) = \lambda^2 \int_a^b K(s,t)g(t)dt$$

throughout present work we denote eigenvalues by λ^2 rather than by $\frac{1}{\lambda^2}$. $g(t)$ is the corresponding eigenfunction. With this notation the eigenvalues λ^2 of the kernel of an integral equation coincide with those of the corresponding differential equation.[1][2]

[1] Kanwal R.P., Linear Integral Equations, Second Edition, 1997
[2] Kythe P.K., Computational Methods for Linear Integral Equations, 2002