# Inner product spaces and minimal values of functionals

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**Abstract**

We consider a real function which depends on the distances between a variable point and the points of a finite subset $A$ of a linear normed space $X$. We show that $X$ is an inner product space if this function attains its local minimum on a barycenter of points of $A$ with well-chosen weights. Our result generalizes classical results about characterization of inner product spaces and answers a question of R. Durier, which was posed in his article [J. Math. Anal. Appl. 207 (1997) 220–239]. 

Let $X$ be a normed linear space and let $S(X)$ be a set of points of norm one. Before we formulate our theorem let us introduce the following:

**Definition.** Let $X$ be a real linear space and $f$ be a functional on X. We say that $x_0 \in X$ is a point of a weak local minimum of the functional $f$ , if for any $y \in X$, there exists $> 0$ such that $f(x_0 + ty) \geq f(x_0)$ for all $t$, $|t| < \varepsilon$.

**Theorem.** Let X be a real normed space, $\dim X \geq 2$ and $n$ be a natural number; $n \geq 3$. Let also $\phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $1 \leq i \leq n$ and $\gamma : \mathbb{R}_+^n \to \mathbb{R}_+$ are some given functions. Consider the statements:

(i) $X$ is an inner product space.

(ii) For every subset $\{a_1, a_2, ..., a_n\}$ included in $S(X)$ and such that $\sum_{i=1}^{n-1} a_i \neq 0$ and $\sum_{i=1}^{n-1} a_i + \|\sum_{i=1}^{n-1} a_i\| a_n = 0$, 0 is the point of a week local minimum of the functional

$$F(x) = \sum_{i=1}^{n-1} \varphi_i(\|x - a_i\|) + \left\| \sum_{i=1}^{n-1} a_i \right\| \varphi_n(\|x - a_n\|).$$

(iii) For every subset $\{a_1, a_2, ..., a_n\}$ included in $S(X)$, for every positive $n$ and every family of real numbers $(\omega_1, \omega_2, ..., \omega_n)$ such that $\sum_{i=n}^{n} \omega_i a_i = 0$, 0 is the point of a weak local minimum of the functional

$$F(x) = \sum_{i=1}^{n} \omega_i \varphi_i(\|x - a_i\|).$$

(iv) For every subset $\{a_1, a_2, ..., a_n\}$ included in $X \backslash \{0\}$ such that $\sum_{i=1}^{n-1} a_i = 0$, 0 is the point of weak local minimum of the functional

$$F(x) = \sum_{i=1}^{n} \|a_i\|^2 \varphi\left(\frac{\|x - a_n\|}{\|a_i\|}\right)$$

(v) for every subset $\{a_1, a_2, ..a_n\}$ from $S(X)$, containing at least one non-collinear vectors and such that $\sum_{i=1}^{n-1} a_i \neq 0$, $\sum_{i=1}^{n-1} a_i \| \sum_{i=1}^{n-1} a_i \| a_n = 0$, 0 is the point of a weak local minimum of the functional

$$F(x) = \gamma\left(\varphi_1(\|x - a_1\|), ..., \varphi_{n-1}(\|x - a_{n-1}\|), \left\| \sum_{i=1}^{n-1} a_i \right\| \varphi(\|x_n - a_n\|)\right)$$

(vi) for every subset $\{a_1, a_2, ..a_n\}$ from $S(X)$, containing at least one pair of non-collinear vectors and $a_n \in X \backslash \{0\}$, such that $\sum_{i=1}^{n} a_i = 0$, 0 is the point of weak local minimum of functional

$$F(x) = \gamma\left(\varphi_1(\|x - a_1\|), ..., \varphi_{n-1}(\|x - a_{n-1}\|), \|a_n\|^2 \varphi_n\left(\frac{\|x_n - a_n\|}{\|a_n\|}\right)\right)$$

The following implications are valid:

(i) If $\phi_i$,$1 \leq i \leq n$, is the function defined on the neighborhood $U$ of the point 1 in $\mathbb{R}$ such that $\varphi_i'(u)$ is continuous, $\varphi_1'(1) = ... = \varphi_n'(u) \neq 0$ and $\varphi_i''(1) > 0$, then $(i) - (iv)$ are equivalent.

(ii) If $\gamma$ is the function defined on the neighborhood of a line $T = (\varphi_1(1), ..., \varphi_{n-1}(1), t_n \phi_n(1))$, such that it has continuous partial derivatives and $\gamma_{t_1}'(t) = ... = \gamma_{t_n}'(t)$ for all $t \in T$ and $\varphi_i$ is a function defined on a neighborhood of a point 1, $\varphi_i'(u)$, $1 \leq i \leq n$, is continuous and $\varphi_1'(1) = ... = \varphi_n'(u) \neq 0$, then we have $(v) \to (i)$, $(vi) \to (i)$.

In [1, Theorem 5.3] the equivalence of statements $(i)$–$(iv)$ was proved for the case when $\varphi_i(t) = t^\alpha$, $\alpha \geq 1$. One of the question from [1], was to find monotone $n$-norms (i.e., norm on $\mathbb{R}^n$ such that if $0 \leq u_i \leq v_i$ ( $1 \leq i \leq n$ ) than for$u = (u_1, ..., u_n)$, $v = (v_1, ..., v_n)$ we have $\gamma(u) \leq \gamma(v)$) different from $l_\alpha$ norms for which results similar to those given in the above mentioned Theorem 5.3 are true.

**Proof.** We are going to show that (v) → (i), (vi) → (i), (i) → (iv), (i) → (iii).After this the equivalence of (i)–(iv) in the theorem will follow since it is clear that (iii) and (iv) imply (ii) and (ii) implies (v) for the function $\gamma(u) = \sum_i u(i)$.

$(v) \to (i)$ According to the well-known Von Neumann–Jordan criterion it is enough to prove this implication for the case $\dim X = 2$. Thus we should prove that the surface $S(X)$ of the unit ball $B(X)$ in $(R2, \|\cdot\|)$ is an ellipse. The proof is based on the following elementary result from [2] and we give it here for completeness.

**Lemma 1.** There exists an ellipse which is inside the unit ball $B(X)$ and touches $S(X)$ at four points at least.

**Proof.** It is easy to show that an ellipse of maximum area inside $B(X)$ touches $S(X)$ at four points at least (this argument seems to be used frequently, see, e.g., [3, p. 322]).

**Lemma 2.** Let $\varphi$ and $\psi$ be two functions defined on the interval $I = (a - \varepsilon, a + \varepsilon)$, $\varepsilon > 0$, such that $\psi(x) \geq \varphi(x), \in I$, $\psi(a) = \varphi(a)$ and the derivatives $\varphi'(a)$, $\psi'_-(a)$, $\psi'_+(a)$ exist. If $\psi'_-(a) \geq \psi'_+(a)$, then $\psi'_-(a) = \psi'_+(a) = \varphi'(a)$.

**Proof.**

$$\varphi'(a) = \lim_{u \to 0, u > 0} \frac{\varphi'(a) - \varphi'(a-u)}{u} \geq \lim_{u \to 0, u > 0} \frac{\psi'(a) - \psi'(a-u)}{u}$$

$$= \psi{-}'(a) \geq \psi{+}'(a) = \lim_{u \to 0, u > 0} \frac{\psi'(a+u) - \psi'(a)}{u} \geq \lim_{u \to 0, u > 0} \frac{\varphi'(a+u) - \varphi'(a)}{u}$$

which proves the lemma.

Let E be the ellipse from Lemma 1 and $A'$ and $B'$ be the points of the intersection $S(X)\cap E$, $A'\neq B'$, and $A'\neq -B'$. Apply an affine transformation $L$ that carries $E$ into the unit circle of $\big(\mathrm{R}^2, \|\cdot\|_2\big)$ $\big(\|\cdot\|_2$ being the usual $l_2$ norm$\big)$. Let $XOY$ be an orthogonal Cartesian system on $\mathbb{R}^2$ such that $L(A')=(-1,0)$. Denote $(-1,0)$ by $A$ and $L(B')$ by $B=(b_1,b_2)$. Obviously, $b_1^2+b_2^2=1$ and $b_2\neq 0$.

Let $a_1=\ldots=a_{n-2}=A$, $a_{n-1}=B$, $a_n'=-\big((n-2)a_1+a_{n-1}\big)$, $a_n=\frac{a_n'}{\|a_n'\|}$, and let $M_\varepsilon$ be the point $M_\varepsilon=(a\varepsilon,\varepsilon)$ , $a=\frac{x}{y}$, where $a_n=(x,y)$. From $b_2\neq 0$ follows that $y\neq 0$. Consider the vectors

$$a_1-M_\varepsilon=(-1-a\varepsilon,\varepsilon),\quad a_{n-1}-M_\varepsilon=(b_1-a\varepsilon,b_2-\varepsilon)$$

$$a_n-M_\varepsilon=(x-a\varepsilon,y-\varepsilon)$$

Since $x=ay$ we get

$$a_n-M_\varepsilon=(ay-a\varepsilon,y-\varepsilon)=\frac{y-\varepsilon}{y}(x,y)$$

and hence $\|a_n-M_\varepsilon\|=1-\frac{\varepsilon}{y}$. We are going to estimate the norms of the two other vectors. By Lemma 2 there exists the tangents to $L(S(X))$ at the points $A$ and $B$ and they are expressed by the equations $x=-1$, $y=-b(x-b_1)+b_2$, respectively, where $b=\frac{b_1}{b_2}$. We may assume that $L(S(X))$ coincides to those tangents at the neighborhood of the points $A$ and $B$, so we have the following expressions:

$$\|a_1-M_\varepsilon\|=1+a\varepsilon+o(\varepsilon)$$

and

$$\|a_{n-1}-M_\varepsilon\|=1-(b_2+ab_1)\varepsilon+o(\varepsilon)$$

By the property of the functional $F$, there exists $\varepsilon>0$ such that for all $\varepsilon$, $|\varepsilon|<\varepsilon$. For such $\varepsilon$ we have

$$F(x)=\gamma\Bigg(\varphi_1(\|M_\varepsilon-a_1\|,\ldots,\varphi_{n-1}(\|M_\varepsilon-a_{n-1}\|,\|a_n'\|\varphi_n(\|M_\varepsilon-a_n\|)\Bigg)$$

$$=\gamma\Bigg(\varphi_1(1+a\varepsilon+o(\varepsilon)),\ldots.,\varphi_{n-2}(1+a\varepsilon+o(\varepsilon)),$$

$$\varphi_{n-1}(1-(b_2+ab_1)+o(\varepsilon)),\|a_n'\|\varphi_n\Big(1-\frac{\varepsilon}{y}\Big)\Bigg).$$

Using Taylor decomposition for $\varphi_i$, $i=1,\ldots,n$, and $\gamma$, we obtain

$$F(M_\varepsilon)=\gamma\Bigg(\varphi_1(1)+\varphi_1'(1)a\varepsilon+o(\varepsilon),\ldots,\varphi_{n-1}(1)-\varphi_{n-1}'(1)(b_2+ab_1)\varepsilon+o(\varepsilon),$$

$$\|a_n'\|\Big(\varphi_n(1)-\varphi_n'(1)\frac{\varepsilon}{y}+o(\varepsilon)\Big)\Bigg).$$

Introduce the notation $\bar{t}=\big(\varphi_1(1),\ldots,\varphi_{n-1}(1),\|a_n'\|\varphi_n(1)\big)$. Since $\gamma_{t_1}'(\bar{t})=\ldots=\gamma_{t_n}'(\bar{t})$ and $\varphi_1'(1)=\ldots=\varphi_n'(1)$ we can rewrite the last equation as follows:

$$F(M_\varepsilon)=\gamma(\bar{t})+\gamma_{t_1}'(\bar{t})\varphi_1'(1)\Bigg((n-2)-(b_2-ab_1)-\frac{\|a_n'\|}{y}\Bigg)\varepsilon+o(\varepsilon)$$

$$\geq \gamma(\bar{t}) = F(0).$$

From this inequality we get

$$\gamma'_{t_1}(\bar{t})\varphi'_n(1)\left((n-2)a - (b_2 + ab_1) - \frac{\|a'_n\|}{y}\right) = 0$$

and since $\gamma'_{t_1}(\bar{t})\varphi'_n(1) \neq 0$ we obtain

$$y = \frac{\|a'_n\|}{(n-2)a - b_2 - ab_1}. \tag{1}$$

using relation $x^2 = a^2y^2$, $\frac{-(n-2-b_1)}{b_2} = a$ and $\|a'_n\| = \frac{-b_2}{y}$, we get

$$x^2 + y^2 = (1+a^2)y^2 = \frac{(1+a^2)b_2}{b_2 + ab_1 - (n-2)a} = \frac{1+a^2}{1+a^2} = 1$$

Denote by $arc(A, B)$ the part of the circle $L(E)$ which is inside smaller angle generated by the vectors $A$ and $B$. As we have just proved, if $L(S(X))$ and $L(E)$ coincide at two points $A$ and $B$ they coincide at one more point $C \in arc(A, B)$. Continuing this process, we see that $L(S(X))$ and arc$(A, B)$ coincide on a dense set of points. Hence $arc(A, B) \subset L(S(X))$ as well. The proof of implication is complete.

(vi) $\to$ (i) Let $A$ and $B$ be the vectors we have just considered above and let $a_1 = \ldots = a_{n-2} = A$, $a_{n-1} = B$, $a_n = -((n-2)a_1 + a_{n-1})$,

$$M_\varepsilon = (a\varepsilon, \varepsilon), a = \frac{x'}{y'}.$$

where $a_n = (x', y')$. Since $x' = ay'$ we get

$$a_n - M_\varepsilon = (ay' - a\varepsilon, y' - \varepsilon) = \frac{y' - \varepsilon}{y'}(x', y')$$

and hence $\frac{\|a_n - M_\varepsilon\|}{\|a_n\|} = 1 - \frac{\varepsilon}{y'}$. It is clear that the same expressions are true for $\|a_1 - M_\varepsilon\|$ and $\|a_{n-1} - M_\varepsilon\|$, so as in the previous case we can derive the equality

$$(n-2)a - (b_2 - ab_1) - \frac{\|a_n\|^2}{y'} = 0.$$

Denote now by $(x, y)$ the vector $\frac{a_n}{\|a_n\|}$, i.e., we have $\frac{y'}{y} = \|a_n\|$. This gives us the equality (1) and hence the relation $x^2 + y^2 = 1$. Using the same arguments as for the previous case we obtain that $L(S(X))$ is a circle. The proof (vi) $\to$ (i) is complete.

(i) $\to$ (vi) let $x \in X$, $\|x\| = \varepsilon$. It is clear that

$$\frac{\|x - a_i\|}{\|a_i\|} = \sqrt{1 - \frac{2(x, a_i)}{\|a_i\|^2} + \frac{\varepsilon^2}{\|a_i\|^2}}.$$

Denoting $\frac{(2(x,a_i) - \varepsilon^2)}{\|a_i\|^2}$ by $\delta_i$ and using the formula

$$\sqrt{1 - \delta_i} = 1 - \frac{1}{2}\delta_i - \frac{1}{8}\delta_i^2 + o(\delta_i^2).$$

we get

$$\sum_{i=n}^{n} \|a_i\|^2 \varphi_i\left(\frac{\|x-a_i\|}{\|a_i\|}\right) = \sum_{i=n}^{n} \|a_i\|^2 \varphi_i\left(1 - \frac{1}{2}\delta_i - \frac{1}{8}\delta_i^2 + o(\delta_i^2)\right).$$

Let

$$\delta_i' = -\frac{1}{2}\delta_i - \frac{1}{8}\delta_i^2 + o(\delta_i^2).$$

Since $\varphi_i''(t)$ is continuous in the neighborhood of the point 1 we have

$$\sum_{i=n}^{n} \|a_i\|^2 \left(\varphi_i(1) + \varphi_i'(1)\delta_i' + \frac{1}{2}\varphi_i''(1)\delta_i'^2 + o(\delta_i'^2)\right).$$

The first term of this expression is $F(0)$. Consider the second one:

$$\sum_{i=n}^{n} \|a_i\|^2 \varphi_i' \delta_i'$$

$$= \varphi_i' \sum_{i=n}^{n} \|a_i\|^2 \left(\frac{\left(-2(x,a_i) - \varepsilon^2\right)}{2\|a_i\|^2} - \frac{1}{8}\left(\frac{2(x,a_i) - \varepsilon^2\varepsilon^2}{\|a_i\|^2} + o(\varepsilon^2)\right)\right.$$

$$\varphi_i'\left(-\left(x, \sum_{i=1}^{n} a_i\right) + \frac{n}{2}\varepsilon^2 - \frac{1}{2}\sum_{i=1}^{n} \frac{(x,a_i)^2}{\|a\|^2} + o(\varepsilon^2)\right).$$

For the third term we have

$$\frac{1}{2}\sum_{i=n}^{n} \|a_i\|^2 \varphi_i''(1)\delta_i'^2 = \frac{1}{2}\sum_{i=n}^{n} \varphi_i''(1)\frac{(x,a_i)^2}{\|a_i\|^2} + o(\varepsilon^2).$$

Since $\sum_{i=n}^{n} a_i = 0$, it is easy to obtain that

$$F(x) = F(0) + \frac{n}{2}\varepsilon^2\varphi_i'(1) + \frac{1}{2}\sum_{i=1}^{n} \frac{(x,a_i)^2}{\|a_i\|^2}\left(\varphi_i''(1) - \varphi_1'(1)\right) + o(\varepsilon^2)$$

$$\geq F(0) + \frac{1}{2}c\varepsilon^2 + o(\varepsilon^2),$$

where $c = \min_{1 \leq i \leq n}\left(\varphi_i''(1), \varphi_1'(1)\right) > 0$. The proof of this implication is complete.

(iii) $\rightarrow$ (iii) For $\|x\| = \varepsilon$ we have $\|x - a_i\| = \sqrt{1 - 2(x,a_i) + \varepsilon^2}$. Denoting $2(x,a_i) - \varepsilon^2$ by $\delta_i$ we get

$$\sum_{i=1}^{n} \omega_i \varphi_i\left(\|x - a_i\|\right) = \sum_{i=1}^{n} \omega_i \varphi_i\left(1 - \frac{1}{2}\delta_i - \frac{1}{8}\delta_i^2 + o(\delta_i^2)\right).$$

As in the previous case we can derive that

$$F(x) = F(0) + \frac{1}{2}\sum_{i=1}^{n} \omega_i\left(\varepsilon^2\varphi_i'(1) + (x,a_i)^2\left(\varphi_i''(1) - \varphi_i'(1)\right)\right) + o(\varepsilon^2)$$

$$\geq F(0) + \frac{1}{2}c\varepsilon^2 + o\big(\varepsilon^2\big).$$

The proof of the theorem is complete.

Now it is easy to find monotone norms different from $l_\alpha$ norms for which the coincidence of optimal location and barycenters of a finite set implies that $X$ is an inner product space. For example, let us consider the following norm $\gamma$ on $\mathbb{R}^n$:

$$\gamma(u_1, ..., u_n) = \sqrt{(n-1)(u_1^2 + ... + u_n^2)} + |u_n|$$

and let $\varphi_t(t) = t\ 1 \leq i \leq n$. We now have

**Proposition.** The following statements are equivalent:

(i) $X$ is an inner-product space.

(ii) For every subset $\{a_1, a_2, ..., a_n\}$ from $S(X)$ containing at least on pair of non-collinear points and such that $\sum_{i=1}^{n-1} a_i \neq 0$, $\sum_{i=1}^{n-1} a_i + \| \sum_{i=1}^{n-1} a_i \| a_n = 0$, 0 is the point of a weak local minimum of the functional

$$F(x) = \sqrt{(n-1)\big(\|x-a_1\|^2 + ... + \|x - a_{n-1}\|^2\big)} + \left\| \sum_{i=1}^{n-1} a_i \right\| \|x - a_n\|$$

(iii) for every subset $\{a_1, a_2, ..a_n\}$ from $S(X)$, containing at least one pair of non-collinear vectors and $a_n \in X \backslash \{0\}$, such that $\sum_{i=1}^{n} a_i = 0$, 0 is the point of weak local minimum of functional and $\sum_{i=1}^{n-1} a_i + \| \sum_{i=1}^{n-1} a_i \| a_n = 0$, 0 is the point of a week local minimum of the functional

$$F(x) = \sqrt{(n-1)\big(\|x-a_1\|^2 + ... + \|x - a_{n-1}\|^2\big)} + \|a_n\| \|x - a_n\|.$$

**Proof.** It is obvious that *gamma* and for all $u = (1, ..., 1, u_n)$, $u_n > 0$, $\gamma'_{u_1}(u) = ... = \gamma'_{u_1}(u) = 1$ i.e., conditions of $\big(\text{v}\big)$ and $\big(\text{vi}\big)$ from the previous theorem hold and we obtain $\big(\text{iii}\big) \to \big(\text{i}\big)$, $\big(\text{ii}\big) \to \big(\text{i}\big)$. Now we will prove that $\big(\text{i}\big) \to \big(\text{ii}\big)$. Let $x \in X$, $\|x\| = \varepsilon$. It is clear that $F(0) = n - 1 + \| \sum_{i=1}^{n-1} a_i \|$ and

$$F(x) = (n-1)\sqrt{1 - \left(\frac{2}{n-1}\left(x, \sum_{i=1}^{n-1} a_i\right) - \varepsilon^2\right)} + \| \sum_{i=1}^{n-1} a_i \| \sqrt{1 - \big(2(x, a_n) - \varepsilon^2\big)}$$

Using formula

$$1 - \delta = 1 - \frac{1}{2}\delta - \frac{1}{8}\delta^2 + o(\varepsilon^2),$$

we get,

$$F(X) = (n-1)\left(1 - \frac{1}{2}\left(\frac{2}{n-1}\left(x, \sum_{i=1}^{n-1} a_i\right) - \varepsilon^2\right) - \frac{1}{8}\left(\frac{2}{n-1}\left(x, \sum_{i=1}^{n-1} a_i\right) - \varepsilon^2\right)^2\right)$$

$$+ \left\| \sum_{i=1}^{n-1} a_i \right\| \left(1 - \frac{1}{2}\big(2(x, a_n) - \varepsilon^2\big) - \frac{1}{8}\big(2(x, a_n) - \varepsilon^2\big)^2\right) + o\big(\varepsilon^2\big).$$

Using condition $\sum_{i=1}^{n-1} a_i + \| \sum_{i=1}^{n-1} a_i\| a_n = 0$, we obtain

$$F(x) = F(0) + \frac{1}{2}(n-1)\varepsilon^2 - \frac{1}{2(n-1)}\left(x, \sum_{i=1}^{n-1} a_i\right)^2$$

$$+\left(\frac{1}{2}\varepsilon^2 - \frac{1}{2}(x, a_n)^2\right)\left\|\sum_{i=1}^{n-1} a_i\right\| + o(\varepsilon^2)$$

Since the set $\{a_1, ...a_{n-1}\}$ contains at least one pair of non-colinnear points there exists $c < 1$ such that $\|\sum_{i=1}^{n-1} a_i\| = c(n-1)$. This gives us the following inequality:

$$\left(x, \sum_{i=1}^{n-1} a_i\right)^2 < \varepsilon^2(n-1)^2c^2$$

and we get

$$F(x) \geq F(0) + \frac{1}{2}\varepsilon^2(n-1)(1-c^2) + o(\varepsilon^2)$$

we can rewrite this as follows:

$$\frac{F(x) - F(0)}{\varepsilon^2} \geq \frac{1}{2}(n-1)(1-c^2) + \frac{o(\varepsilon^2)}{\varepsilon^2}$$

i.e., we obtain that 0 is the point of a local minimum and hence the weak local minimum of the functional $F(x)$. The proof of this implication is complete. Implication $(\mathrm{i}) \to (\mathrm{iii})$ can be proved similarly. The proof of the proposition is complete.

**References.**


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